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On Discounting and Voting in a Simple Growth Model*

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Abstract

In dynamic resource allocation models, the non-existence of voting equilibria is a generic phenomenon due to the multi-dimensionality of the choice space even if agents are heterogeneous only in their discount factors. Nevertheless, at each point in time there may exist a "median voter" whose preferred instantaneous consumption rate is supported by a majority of agents. Based on this observation, we propose an institutional setup ("intertemporal majority voting") in a Ramsey-type growth model with common consumption and heterogeneous agents, and show that it provides a microfoundation of the choice of the optimal consumption stream of the "median" agent. While the corresponding intertemporal consumption stream is in general not a Condorcet winner among all feasible paths, its induced instantaneous consumption rates receive a majority at each point in time in the proposed intertemporal majority voting procedure. We also provide a characterization of stationary voting equilibria in the case where agents may differ not only in their time preferences, but also in their felicity functions.

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1 Introduction

It is well-known that in multi-dimensional models a voting equilibrium under majority rule fails to exist generically (Plott, 1967; McKelvey, 1976). This holds in particular in

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dynamic multi-period resource allocation problems — even if choices at different points in time are linked by an intertemporal resource constraint (Boylan et al., 1996), and even in cases where in addition the agents' type space is one-dimensional (see, e.g., De Donder et al., 2012). One-dimensional type spaces arise naturally when agents differ only in discount factors, a case that has received considerable attention recently (see, e.g., Heal and Millner, 2014; Jackson and Yariv, 2015).

Notwithstanding these negative results, at each point in time there may exist a "median voter" whose preferred choice of instantaneous consumption rate is supported by a majority of agents. Based on this observation, we propose a simple institutional setup in a Ramsey-type growth model, *intertemporal* majority voting, that does not suffer from the problem of generic non-existence of equilibrium. Importantly, in this setup the temporary voting is not over consumption *levels* but over consumption *rates*, since the latter variable entails a degree of freedom given the intertemporal resource constraint and the decisions (on consumption rates) in all other periods. The equilibrium concept that we employ is Kramer–Shepsle equilibrium with perfect foresight; that is, (i) each period's decision follows the (sincere) majority vote under the assumption that agents maximize their utility given the outcomes of all other periods, and (ii) agents' expectations about these outcomes are correct in equilibrium ("perfect foresight"). It is worth emphasizing that the institutional framework is well-defined also without the assumption of perfect foresight.

We show that if agents have the same felicity function and differ only in their discount factors, the outcome of intertemporal majority voting (an *intertemporal voting equilibrium*) is unique and coincides with the optimal consumption stream of the agent with the median discount factor. As an important intermediate result we establish that, for each fixed agent, the step-by-step determination of the optimal consumption rate under perfect foresight yields the optimal overall intertemporal consumption stream. While this technical result probably belongs to the body of "folk wisdom" within the Ramsey model, we are not aware of a rigorous proof and provide one here. We also consider the multi-dimensional heterogeneity case in which agents differ both in their felicity functions and discount factors. For this general case we provide a characterization of steady-state and balanced-growth voting equilibria.

1.1 Common property resource problems

The problem of aggregating heterogeneous time preferences arises in many contexts.[1] The model we propose in this paper admits two interpretations. It can be viewed either as a Ramsey-type model of growth and physical (man-made) capital accumulation, or as a model of renewable or exhaustible natural resource allocation over time.

Though we present our model in terms of the traditional theory of economic growth, it is instructive to consider the problem of aggregating heterogeneous time preferences also within the common property resource framework. Examples are the hunting for animals, the grazing of cattle on a common ground, the pollution of the atmosphere, or the drilling for oil in a common underground reservoir.

In these contexts, an issue of evident importance is the determination of the socially desirable harvest (extraction) rate. Consider a village situated near a fishing ground. The fishing ground is self-managed by village citizens, who differ in their time preferences. The

[1]For recent evidence that individuals indeed differ in their discount factors see, e.g., Wang et al. (2010), Castillo et al. (2011), and Schaner (2015).

question is: what is the harvest rate of the fish stock collectively set by heterogeneous agents? If all citizens in the village were identical, then the rate of the fish stock exploitation could be easily determined using their common discount factor. However, it is not clear how to determine the harvest rate when citizens have different discount factors.

One might try to argue that the introduction of property rights can (indirectly) solve the problem. Indeed, the typical and well-known solution to the "tragedy of the commons" is to establish private property rights. Once the property rights are enforced, each owner acts optimally according to her own time preference. This might circumvent the problem in cases where suitable property rights can be established.[2] However, often the non-excludability of public goods prevents the enforcement of suitable private property rights. This is likely to occur in the case of the underground oil reservoir, the fishing ground, or the so-called "global commons". For instance, the tendency of fish to migrate makes it impossible to define geographically determined property rights over the fish stock. In this case a solution may be to introduce a governmental or community resource ownership, but then it is necessary to find a non-market mechanism of determining the harvest rate.

1.2 Social choice in the optimal growth model

There is a vast amount of literature devoted to the Ramsey general equilibrium models with heterogeneous agents who differ in their discount factors (see Becker, 2006, for an excellent survey). In this kind of models each agent separately solves her own optimization problem and thus has an independent private consumption stream.

However, the Ramsey framework also allows one to study how heterogeneous agents make joint decisions over common consumption streams. Note that it does not matter whether "common consumption" is a collectively consumed public good or a private good that is consumed according to some fixed and commonly known sharing rule. What is important is that agents' personal utilities are based on their collective decisions, i.e., on the common consumption stream they choose. Here, economic growth theory meets social choice theory, and there is indeed a literature that analyzes how political institutions can be incorporated into growth models in order to determine collective choices among heterogeneous agents (see, e.g., Beck, 1978; Boylan, 1995; Boylan et al., 1996).

A natural way of aggregating heterogeneous preferences is voting. Suppose that agents vote over all feasible consumption streams by pairwise majority voting. Then, it is well-known that, due to the high dimensionality of the underlying choice space, there does not in general exist a Condorcet winner, i.e., for every feasible consumption stream there exists another feasible consumption stream that is preferred by a majority (see, e.g., Davis et al., 1972; Kramer, 1973; Bucovetsky, 1990).[3] Moreover, the fact that agents differ only in one parameter does not help: there still is no Condorcet winner in voting over a multi-dimensional choice space even if the agents' type space is one-dimensional (see, e.g., De Donder et al., 2012). Boylan et al. (1996) consider voting over feasible consumption paths in the Ramsey optimal growth model and prove that there is in general no Condorcet winner. In a more recent paper, Jackson and Yariv (2015) analyze the case of agents who differ only in their discount factor, and prove a general impossibility theorem which implies, among other things, that any non-dictatorial aggregation rule of feasible consumption plans satisfying mild further conditions admits cyclical social preferences.

[2]If the "owners" are *groups* of individuals with heterogeneous time preferences, the problem might of course persist within these groups.

[3]Bernheim and Slavov (2009) characterize this kind of situation as the "curse of dimensionality".

Despite these negative results, it appears that in a model in which agents differ only in their discount factors, the optimal consumption path of the agent with the *median* discount factor has some claim for being a natural and appealing collective choice. But clearly, the mentioned impossibility results imply in particular that also the optimal consumption path for the "median" agent is in general not a Condorcet winner among all feasible paths.

One way to overcome this difficulty has been considered by Beck (1978) and, more recently, by Heal and Millner (2014). In these models, agents are only allowed to vote over the set of individually optimal paths. It can be shown that, among all individually optimal paths, the optimal path for the agent with the median discount factor is indeed a Condorcet winner. However, ensuring the existence of a stable voting outcome in this way is not very satisfactory since it is made possible only by severely restricting the choice set.

A different voting mechanism is proposed by Boylan et al. (1996) who introduce two additional agents ("political candidates") to the model. In each period the candidates propose a consumption level for the agents and care only about being elected. Agents vote for one of the candidates and care only about consumption. A specific noncooperative game is then constructed, and it is shown that the subgame perfect equilibrium coincides with the optimal path for the agent with the median discount factor. Although it yields the desired and intuitive outcome, this voting procedure seems quite contrived and complex.

The purpose of the present paper is to propose a more intuitive and tractable voting procedure that yields as outcome the optimal consumption path of the agent with the median discount factor if agents have the same felicity function, and that can be applied also in the general case in which agents have different discount factors and different felicity functions.[4]

1.3 Intertemporal majority voting

We consider a Ramsey-type growth model with agents who may differ in their instantaneous utility functions ("felicity functions") and time preferences. Agents maximize their intertemporal discounted utilities by allocating at each point in time a given amount of a single good between consumption which provides instant utility, and investment which is used in production. The technology is described by a production function, which is assumed to be either neoclassical or linear.

We suppose that agents share a common consumption stream. The good is either consumed collectively or privately according to some fixed sharing rule. In the common property resource interpretation of our model, the capital stock is viewed as the renewable resource stock, the production function becomes the regeneration function, and the consumption level is the amount of the resource extracted (= the harvest rate times the available resource stock).

Within our framework, we propose a simple and natural voting procedure according to which agents choose a consumption path from the set of all feasible consumption paths by "intertemporal majority voting". The two crucial principles in this institutional setup

[4]The idea to use dynamic voting in order to determine a stable outcome has been investigated in a number of specific models. In Borissov et al. (2014a) agents vote for a tax aimed at environmental maintenance; Borissov et al. (2014b) study voting over the shares of public goods in GDP, and Borissov and Pakhnin (2016) consider voting over extraction rates in a model with exhaustible natural resources. In all cases, the equilibrium policy is determined by the agent with the median discount factor as in the present paper. However, these models lack generality because they use specific forms of the utility and production functions.

are that (i) voting is done "step-by-step", and (ii) voting is not over the consumption level itself, but over the consumption *rate*. We avoid the "curse of dimensionality" by transforming a multi-dimensional choice space into a series of one-dimensional choice spaces. Indeed, the dynamic intertemporal structure of the model naturally suggests to consider institutions that also allow for intertemporal choices of agents. The solution concept given the proposed intertemporal voting procedure is *Kramer–Shepsle* equilibrium (Kramer, 1972; Shepsle, 1979).[5]

Given the general idea to transform the multi-dimensional choice problem into a sequence of one-dimensional choice problems, an important issue that has to be addressed is how the expectations should be formed. The important feature of our approach is that expectations are formed precisely about future consumption *rates*. If this is the case, agents can vote today over the consumption level as well as over the consumption rate, the outcome of voting will be the same. On the other hand, one-dimensional voting over the current consumption level under given expectations about the consumption *levels* in all other periods is pointless, since consumption in each period is uniquely determined by consumption in all other periods via the overall resource constraint. Thus, if future consumption is given, there is no trade-off between consumption today and consumption in the future (we provide a simple example in Subsection 3.1 below to illustrate this point), and formally, *every* feasible consumption path is a Kramer–Shepsle equilibrium in our model.

To implement the idea of intertemporal majority voting in a fruitful manner, we look at the problem at hand from a slightly different perspective. Originally, the model is formulated in terms of consumption *levels*, and mathematically, it involves an optimal control problem with the consumption level as control variable. We make a change of variables and use instead the consumption *rate* (i.e., $1 -$ savings rate) as control variable.[6] Using this change of variable, we define a voting equilibrium in two stages, following the traditions of dynamic macroeconomics and applying the Hicks–Grandmont temporary equilibrium approach (Hicks, 1939; Grandmont, 1977).

First, for any point in time agents vote by majority rule over the current consumption rate, given the current capital stock and *some* expectations about future consumption rates. This yields a one-dimensional decision problem, and we show that agents' preferences over the current consumption rate are single-peaked, and therefore the median voter theorem applies. If, in addition, agents have the same felicity function and the same expectations, at each given point in time the temporary voting equilibrium (i.e., the instantaneous Condorcet winner) is the preferred consumption rate for the agent with the median discount factor.

Second, the intertemporal majority voting equilibrium is defined as a sequence of temporary voting outcomes such that all agents have *perfect foresight* about future outcomes of voting. We prove that if agents have the same felicity function there is a unique intertemporal voting equilibrium, which is the optimal consumption path for the agent with the median discount factor. The proof is based on the general result that the step-by-step

[5]In general, a vector of policies is a Kramer–Shepsle equilibrium if, for any single dimension, the corresponding policy in this dimension coincides with the majority choice, given the equilibrium choices in all other dimensions. Clearly, if the multi-dimensional problem admits a Condorcet winner, then the Condorcet winner constitutes a Kramer–Shepsle equilibrium.

[6]The savings rate as control variable in the Ramsey model has been used by Phelps and Pollak (1968) and Peleg and Yaari (1973). These authors study agents' behavior under time-inconsistent preferences, and ask when the chosen plan of actions will be actually followed by rational individuals in the future. By contrast, we assume time-consistent preferences throughout.

determination of the consumption rate under perfect foresight for any given agent results in the ("once-and-for-all") optimum in terms of consumption levels for this agent. This result, though not surprising, is of interest in itself, and we present it in Section 5 below.

We thus view our analysis as providing an institutional "microfoundation" for the choice of the optimal consumption path for the agent with the median discount factor, a proposal that has been repeatedly put forward in the literature but without an ultimately appealing justification so far.

If agents differ both in their discount factors and felicity functions, the intertemporal voting equilibrium is clearly no longer determined by the discount factor alone. However, even with infinite-dimensional heterogeneity we are still able to obtain some results. In the case of a strictly concave production function, we define a steady-state voting equilibrium, and show that it is unique and again determined by the median discount factor; in the case of a linear production function, we define a balanced-growth voting equilibrium, prove its existence and uniqueness, and show that it is determined by the "median growth rate".

1.4 Further remarks on related literature

The outcome of intertemporal majority voting (i.e., the optimal consumption path of the "median" agent) is clearly both time-consistent and Pareto efficient. This may seem at odds with the result of Jackson and Yariv (2015), who argue that any time-consistent and Pareto efficient voting rule must be dictatorial. There is in fact no contradiction here since the choice of an optimal path of one agent is indeed *ex post* "dictatorial". Note, however, that unlike Jackson's and Yariv's notion, the usual definition of dictatorship in social choice theory is a much stronger notion of *ex ante* dictatorship: an aggregation rule is dictatorial if only one agent is decisive *no matter* what the preferences of all other agents' are. Evidently, any aggregation rule that follows the preferences of the median voter is necessarily "ex post dictatorial", but this does not imply dictatorship in the usual sense of social choice theory. The same holds for the choice of the optimal consumption path of the "median" agent in our institutional setup.

In this paper we do not address questions related to uncertainty of future economic development. There is a lively and ongoing discussion on how to discount the future under uncertainty (see, e.g., Pearce et al., 2003; Gollier and Weitzman, 2010; Traeger, 2013). While the introduction of uncertainty seems to bring the problem of choosing a consumption path closer to real life decisions, it also complicates matters quite dramatically. Our hope is that, even though our analysis does not directly contribute to the literature of discounting the future under uncertainty, the idea of intertemporal majority voting might be also fruitfully applicable to this more general setting.

The rest of the paper is organized as follows. Section 2 introduces our model. In Section 3 two simple examples illustrate the idea of intertemporal voting and explains the role of consumption rate. In Section 4 we define temporary and intertemporal voting equilibria. In Section 5 we consider the step-by-step decision-making process for a single agent. Section 6 states our main results. We prove the existence of temporary equilibria and show that if all agents have the same felicity function and differ only in their time preferences, an intertemporal voting equilibrium exists, is unique and coincides with the optimum in terms of consumption rates for the agent with the median discount factor. In Section 7 we study the general case where agents differ both in their discount factors and felicity functions, and characterize steady-state and balanced-growth voting equilibria. Section 8 concludes.

2 The model

We consider a Ramsey-type growth model with heterogeneous agents and common consumption. Suppose $T \in \mathbb{N} \cup \{\infty\}$ is the length of the time horizon, which can be finite or infinite. Let time be $\mathbb{T} = \{0, 1, \ldots, T\}$ when $T < \infty$, and $\mathbb{T} = \{0, 1, \ldots\}$ when $T = \infty$.

There is an odd number N of heterogeneous agents indexed by $i = \{1, 2, \ldots, N\}$. The heterogeneity is captured by agents' discount factors and felicity functions. Agent i has a discount factor $\delta_i \in (0, 1)$, and a felicity function $u_i : \mathbb{R}_{++} \to \mathbb{R}$ which satisfies the following conditions:

$$u_i'(c) > 0, \quad u_i''(c) < 0, \quad \lim_{c \to 0} u_i'(c) = +\infty.$$

Her intertemporal utility function is of the form $\sum_{t \in \mathbb{T}} \delta_i^t u_i(c_t)$, where $C = \{c_t\}_{t \in \mathbb{T}}$ is the *common* consumption stream. It is not critical whether actual consumption is common or private. In the latter case there is a fixed and commonly known sharing rule. For example, if this rule is egalitarian, then $u_i(c)$ should be replaced with $u_i(c/N)$. What is important is that agents' personal utilities are based on their collective decisions.

A single homogeneous good is produced. In each period $t \in \mathbb{T}$ the available amount of good is allocated between consumption c_t and capital k_{t+1} for use in the next period production: $c_t + k_{t+1} = f(k_t)$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a production function.

As was noted above, our model can also be viewed as a common property resource model. In this case k should be viewed as the stock of a renewable or exhaustible natural resource and $f(k)$ as a function describing regenerative capacity of the resource and consumption as the amount of the resource extracted. In resource economics, the regenerative capacity of a resource is typically described in terms of a so-called regeneration (net growth) function $g(k)$, which gives the level of net growth of the resource stock depending on the size of the stock, k. If the resource stock at the beginning of period t is k_t and harvest during period t is c_t, then the resource stock at the beginning of period $t+1$ is $k_{t+1} = g(k_t) - c_t + k_t$. Thus, if we interpret our model as a common property resource model, then $f(k) = g(k) + k$. If the resource is exhaustible, then $f(k) = k$.

We assume that either

Case 1. ***Strictly concave production function***
The production function satisfies the following properties:

$$f(0) = 0, \quad f'(k) > 0, \quad f''(k) < 0, \quad \exists \bar{k} \; : \; f(\bar{k}) = \bar{k}, \quad \delta_{min} f'(0) > 1, \tag{1}$$

where δ_{min} is the minimal discount factor in the set $\{\delta_i\}_{i=1}^N$

or

Case 2. ***Linear production function***
The production function is linear:

$$f(k) = Ak, \quad A > 0.$$

In this case we additionally assume that the felicity function of every agent is of the CIES (constant intertemporal elasticity of substitution) form:

$$u_i(c) = \begin{cases} \frac{c^{1-\rho_i}}{1-\rho_i}, & \text{if} \quad 0 < \rho_i < +\infty, \; \rho_i \neq 1, \\ \ln c, & \text{if} \quad \rho_i = 1. \end{cases} \tag{2}$$

For each agent i, consider the following optimization problem:

$$\max \sum_{t\in\mathbb{T}} \delta_i^t u_i(c_t), \quad \text{s. t.} \quad c_t + k_{t+1} = f(k_t), \quad c_t \geq 0, \quad k_{t+1} \geq 0, \quad t \in \mathbb{T}. \tag{3}$$

In Case 1 (strictly concave production function), maximization problem (3) has a unique solution (optimal path) for each agent i. The same holds true in Case 2 (linear production function) if $\delta_i A^{1-\rho_i} < 1$.

Definition. *The solution to problem (3),* $\{c_t^{i*}, k_{t+1}^{i*}\}_{t\in\mathbb{T}}$*, is called an optimum in terms of consumption levels for agent i.*

3 Intertemporal voting: examples

If all agents have the same felicity functions and discount factors, they have the same optimal path, so there is no collective choice problem. Which consumption stream will be chosen by a society that consists of heterogeneous agents?

A natural way of aggregating heterogeneous preferences is voting, and one may hope that some form of majority voting would result in the outcome supported by the "median voter". For instance, in the case where agents differ only in their time preferences, the median voter is the agent with the median discount factor. However, Boylan et al. (1996) show that in the latter case the path optimal for the agent with the median discount factor is "blocked" by the coalition consisting of all other agents. Moreover, a Condorcet winner does not in general exist.

A natural idea to overcome the absence of a Condorcet winner is to convert a multi-dimensional choice space, made up of consumption streams, into a series of one-dimensional choice spaces. The basic notion of "coordinate-wise" majority voting was proposed independently by Kramer (1972) and Shepsle (1979). This approach in dynamic models can be interpreted as intertemporal (step-by-step) voting under perfect foresight about outcomes of future votes. In this section we consider two simple examples in order to gain intuition about intertemporal voting procedures and analyze the applicability of such procedures to our model.

The aim of the first example is to explain our approach within a simplest framework. It illustrates the absence of a Condorcet winner and shows that one-dimensional voting over current consumption under given expectations about the future consumption levels is pointless, but if the voting agents have perfect foresight about the future consumption *rates*, then the outcome of intertemporal voting coincides with the optimal consumption stream for the "median" agent. The second example relates intertemporal voting to well-known results for the Ramsey model with Cobb–Douglas production function and log utility.

3.1 Finite horizon example

Consider the following three-period three-agent example: $T = 3$, $N = 3$, $u_i(c) = \ln c$, and $f(k) = k$ (thus we are dealing with an intertemporal cake-eating problem).

First, suppose that the agents vote over the whole consumption stream only once, at time 0. It is easy to note that in this case, the set of alternatives over which they vote is $\mathcal{C} = \{(c_0, c_1, c_2) \in \mathbb{R}^3_+ \mid c_0 + c_1 + c_2 = k_0\}$, the objective function of agent i is $\mathcal{U}^i(c_0, c_1, c_2) = \ln c_0 + \delta_i \ln c_1 + \delta_i^2 \ln c_2$, and problem (3) becomes as follows:

$$\max\ \mathcal{U}^i(c_0, c_1, c_2) \quad \text{s. t.} \quad c_0 + c_1 + c_2 = k_0, \quad c_0 \geq 0, \quad c_1 \geq 0, \quad c_2 \geq 0.$$

The solution to this problem, $\{c_0^{i*}, c_1^{i*}, c_2^{i*}\}$, is given by

$$c_0^{i*} = \frac{k_0}{1 + \delta_i + \delta_i^2}, \quad c_1^{i*} = \frac{\delta_i k_0}{1 + \delta_i + \delta_i^2}, \quad c_2^{i*} = \frac{\delta_i^2 k_0}{1 + \delta_i + \delta_i^2}.$$

As was noted above, it seems natural to conjecture that a Condorcet winner exists and coincides with the solution to problem (3) for the agent with the medial discount factor δ_{med}, i.e., with the triple $\{c_0^*, c_1^*, c_2^*\}$ given by

$$c_0^* = \frac{k_0}{1 + \delta_{med} + \delta_{med}^2}, \quad c_1^* = \frac{\delta_{med} k_0}{1 + \delta_{med} + \delta_{med}^2}, \quad c_2^* = \frac{\delta_{med}^2 k_0}{1 + \delta_{med} + \delta_{med}^2}.$$

However, this conjecture is false. Indeed, the gradient of $\mathcal{U}^i$ at the point (c_0^*, c_1^*, c_2^*) is

$$\nabla \mathcal{U}^i(c_0^*, c_1^*, c_2^*) = \frac{1 + \delta_{med} + \delta_{med}^2}{k_0} \left(1, \frac{\delta_i}{\delta_{med}}, \frac{\delta_i^2}{\delta_{med}^2}\right).$$

Let us compute the inner product of $\nabla \mathcal{U}^i(c_0^*, c_1^*, c_2^*)$ and $z = (1, -2, 1)$:

$$\nabla \mathcal{U}^i(c_0^*, c_1^*, c_2^*) \cdot z = \left(1 - 2\frac{\delta_i}{\delta_{med}} + \frac{\delta_i^2}{\delta_{med}^2}\right) \frac{1 + \delta_{med} + \delta_{med}^2}{k_0} = \left(1 - \frac{\delta_i}{\delta_{med}}\right)^2 \frac{1 + \delta_{med} + \delta_{med}^2}{k_0}.$$

It is positive if $\delta_i \neq \delta_{med}$. It follows that for a sufficiently small perturbation of (c_0^*, c_1^*, c_2^*) in the direction z, we can find a consumption stream (c_0', c_1', c_2') which lies in $\mathcal{C}$ and which the agents with $\delta_i \neq \delta_{med}$ prefer to (c_0^*, c_1^*, c_2^*). This consumption stream has more consumption at time 0 (to satisfy the agent whose discount factor is lower than δ_{med}), more consumption at time 2 (to satisfy the agent whose discount factor is higher than δ_{med}) and less consumption at time 1 (to make it feasible). Since two agents out of three prefer (c_0', c_1', c_2') to (c_0^*, c_1^*, c_2^*), the latter consumption stream is not a Condorcet winner. Moreover, it is possible to show (see Boylan et al., 1996) that no other consumption stream is a Condorcet winner and thus a Condorcet winner does not exist.

Let us now try to find a Kramer–Shepsle equilibrium, i.e., a consumption stream each element of which coincides with the majority choice, given the choices of all other elements.

Suppose that at time 0 agents have some common expectations about future consumption, c_1 and c_2, and vote over the time 0 consumption c_0. The preferred time 0 consumption for agent i is the solution to the following problem:

$$\max_{c_0} \ln c_0, \quad \text{s. t.} \quad c_0 + c_1 + c_2 = k_0.$$

However, this optimization problem is degenerate. The overall resource constraint under given expectations fully predetermines the optimal value of c_0. Moreover, this value is the same for all agents: it is optimal for all agents to consume today as much as possible, given the future consumption profile and the initial amount of capital. The same argument applies to voting over c_1 and c_2. It follows that every consumption stream $\{c_0, c_1, c_2\}$ such that $c_0 < k_0$, $c_1 < k_0 - c_0$, and $c_2 = k_0 - c_0 - c_1$, can be obtained as a result of intertemporal voting over consumption levels under perfect foresight. Such a voting procedure seems to be meaningless.

However, this observation does not invalidate the idea to transform a multi-dimensional choice problem into a sequence of one-dimensional choice problems. To look at the same example from a different perspective, let us formulate the initial problem in terms of *consumption rates* $e_0 = c_0/k_0$, $e_1 = c_1/k_1$, $e_2 = c_2/k_2$, instead of consumption levels $\{c_0, c_1, c_2\}$. Then the utility function of agent i takes the form

$$V^i(e_0, e_1, e_2) = \ln(e_0 k_0) + \delta_i \ln(e_1(1-e_0)k_0) + \delta_i^2 \ln(e_2(1-e_1)(1-e_0)k_0),$$

the problem of utility maximization for agent i becomes

$$\max V^i(e_0, e_1, e_2), \quad \text{s. t.} \quad 0 \leq e_0 \leq 1, \quad 0 \leq e_1 \leq 1, \quad 0 \leq e_2 \leq 1,$$

and its solution $\{e_0^{i*}, e_1^{i*}, e_2^{i*}\}$ is given by

$$e_0^{i*} = \frac{1}{1+\delta_i+\delta_i^2}, \quad e_1^{i*} = \frac{1}{1+\delta_i}, \quad e_2^{i*} = 1.$$

Let us apply the intertemporal majority voting procedure to the problem formulated *in terms of consumption rates.* Agents vote over the current consumption rate under given past consumption rates and expectations about future consumption rates. Suppose that at time 0 expectations about future consumption rates are e_1 and e_2. Then the preferred time 0 consumption rate for agent i, e_0^i, is the solution to the following problem:

$$\max_{0 \leq e_0 \leq 1} V^i(e_0, e_1, e_2).$$

It is not difficult to check that it coincides with the first element of the optimum in terms of consumption rates for agent i: $e_0^i = e_0^{i*}$.[7]

Clearly, the preferences of agents in one-dimensional voting over e_0 are single-peaked, and the preferred values e_0^i, $i = 1, 2, 3$, are decreasing in δ_i. By the median voter theorem, the Condorcet winner in this vote exists. It is equal to the preferred time 0 consumption rate for the agent with the median discount factor, $e_0^* = \frac{1}{1+\delta_{med}+\delta_{med}^2}$.

Now consider voting over the time 1 consumption rate. Agents already know that the time 0 consumption rate is equal to e_0^* and have expectations about the time 2 consumption rate, e_2. Then the preferred time 1 consumption rate for agent i, e_1^i, is the solution to the following problem:

$$\max_{0 \leq e_1 \leq 1} V^i(e_0^*, e_1, e_2).$$

Evidently, the preferred time 1 consumption rate for agent i coincides with the second element of her optimum in terms of consumption rates: $e_1^i = e_1^{i*}$. By the median voter theorem, a Condorcet winner exists and is equal to the preferred time 1 consumption rate for the agent with the median discount factor, $e_1^* = \frac{1}{1+\delta_{med}}$.

Finally, the problem of finding the preferred time 2 consumption rate for agent i takes the form:

$$\max_{0 \leq e_2 \leq 1} V^i(e_0^*, e_1^*, e_2).$$

The solution to this problem coincides with $e_2^{i*} = 1$. Since all agents vote unanimously, a Condorcet winner exists and is equal to $e_2^* = 1$.

[7]Due to the simplicity of the example, e_0^i does not depend on expectations about future consumption rates.

Thus we obtain the sequence of consumption rates $E^* = \left\{\frac{1}{1+\delta_{med}+\delta_{med}^2}, \frac{1}{1+\delta_{med}}, 1\right\}$. Each element of E^* is the Condorcet winner in one-dimensional voting over the single consumption rate at the corresponding instant in time under known values of previous consumption rates and given expectations about future consumption rates. It is clear that the sequence E^* is the solution to the utility maximization problem in terms of consumption rates for the agent with the median discount factor.

3.2 Infinite horizon example

Consider now the infinite horizon problem with the same logarithmic felicity function for all agents and a Cobb–Douglas production function. Given $k_0 > 0$, the optimization problem *in terms of consumption levels* for agent i is as follows:

$$\max \sum_{t=0}^{\infty} \delta_i^t \ln c_t, \quad \text{s. t.} \quad c_t + k_{t+1} = k_t^\alpha, \quad c_t \geq 0, \quad k_{t+1} \geq 0, \quad t = 0, 1, \ldots,$$

where $0 < \alpha \leq 1$. It is well-known that the solution to this problem is characterized by a constant over time savings rate, equal to $\alpha\delta_i$. Therefore the optimal consumption rate is also constant over time and equal to $1 - \alpha\delta_i$. It follows that the optimum in terms of consumption rates for agent i is given by the sequence $\{1 - \alpha\delta_i, 1 - \alpha\delta_i, \ldots\}$.

To describe intertemporal majority voting over consumption rates $\{e_0,\ e_1, \ldots\}$ ($e_t = c_t/k_t^\alpha,\ t = 0, 1, \ldots$), suppose that at an arbitrarily chosen point in time, τ, the stock of capital, $k_\tau > 0$, is given and agents have some expectations about future consumption rates, $\{e_t\}_{t=\tau+1}^\infty$. Then the objective function of agent i in voting over e_τ is given by

$$\begin{aligned}
\ln\left(e_\tau(k_\tau)^\alpha\right) + \delta_i \ln\left(e_{\tau+1}(1-e_\tau)^\alpha(k_\tau)^{\alpha^2}\right) \\
+ \delta_i^2 \ln\left(e_{\tau+2}(1-e_{\tau+1})^\alpha(1-e_\tau)^{\alpha^2}(k_\tau)^{\alpha^3}\right) + \ldots \\
= \ln e_\tau + \alpha\delta_i \ln(1-e_\tau) + \alpha^2\delta_i^2 \ln(1-e_\tau) + \ldots + \Gamma_\tau^i = \ln e_\tau + \frac{\alpha\delta_i}{1-\alpha\delta_i}\ln(1-e_\tau) + \Gamma_\tau^i,
\end{aligned}$$

where

$$\Gamma_\tau^i = \ln\left((k_\tau)^\alpha\right) + \delta_i \ln\left(e_{\tau+1}(k_\tau)^{\alpha^2}\right) + \delta_i^2 \ln\left(e_{\tau+2}(1-e_{\tau+1})^\alpha(k_\tau)^{\alpha^3}\right) + \ldots$$

is a term that depends on k_τ and expectations, but does not depend on the variable over which agents vote. If $0 < e_t < 1,\ t > \tau$, and $0 < \liminf_{t\to\infty} e_t \leq \limsup_{t\to\infty} e_t < 1$, then $-\infty < \Gamma_\tau^i < +\infty$ and hence the objective function of each agent is well-defined. To find her preferred time τ consumption rate, agent i needs to solve the following equation:

$$\frac{d}{de_\tau}\left(\ln e_\tau + \frac{\alpha\delta_i}{1-\alpha\delta_i}\ln(1-e_\tau)\right) = 0.$$

Clearly, the solution to this equation is equal to the optimal consumption rate $1 - \alpha\delta_i$.[8]

[8]Due to the log-linear felicity functions and the Cobb–Douglas production function, the preferred time τ consumption rate for each agent is independent of expectations. In a number of papers (see Borissov et al., 2014a,b; Borissov and Pakhnin, 2016), this fact is used to generalize the considered example to voting in a dynamic general equilibrium framework.

The preferences of agents in one-dimensional voting over the time τ consumption rate are single-peaked and the preferred consumption rates negatively depend on δ_i. Therefore, by the median voter theorem, the winner in majority voting over the time τ consumption rate is the preferred consumption rate for the "median" agent (i.e., the agent with the median discount factor δ_{med}), $1-\alpha\delta_{med}$. If voting takes place at each time, we obtain the sequence $\{1-\alpha\delta_{med}, 1-\alpha\delta_{med}, \ldots\}$, which is exactly the optimum in terms of consumption rates for the agent with the median discount factor.

The above examples illustrate two important aspects of using the consumption rate as the control variable. First, for each agent the sequence of the preferred consumption rates coincides with the optimum in terms of consumption rates. Second, intertemporal majority voting over consumption rates yields, as outcome, the optimum in terms of consumption rates for the agent with the median discount factor.

It is natural to ask whether these results can be generalized to a general Ramsey-type model. The main difficulty is that the preferred time τ consumption rate for each agent is generically a function of all expected future consumption rates. If agents form expectations arbitrarily, there is no reason to expect that a reasonable outcome of step-by-step voting procedure will be obtained. However, we shall show that if agents have perfect foresight about future decisions, then under appropriate assumptions the outcome of such a procedure indeed coincides with the optimum in terms of consumption rates for the "median" agent.

4 Intertemporal voting: definitions

As we have seen, the intertemporal majority voting procedure is based on the two crucial principles: (*i*) voting is done step-by-step, and (*ii*) voting is not over consumption levels, but over consumption rates. In this section we begin by presenting the initial optimization problem in terms of consumption rates, and then give a formal definition of an intertemporal voting equilibrium.

4.1 Optimization problem in terms of consumption rates

The control variable in problem (3) is the consumption level c_t. Let us make a change of variables and take the consumption rate

$$e_t = \frac{c_t}{f(k_t)}, \quad t \in \mathbb{T}, \tag{4}$$

as the control variable.[9] Clearly, to be feasible, the sequence of consumption rates must be such that $0 \le e_t \le 1,\ t \in \mathbb{T}$. If we interpret our model as a model of natural resource allocation over time, then the consumption rate e_t becomes the rate of extraction.

Taking into account (4) and the constraints in (3), we can express the time t consumption and capital stock in terms of k_0 and all previous consumption rates:

$$\begin{cases} c_t = e_t f\left((1-e_{t-1}) f\left((1-e_{t-2}) f(\ldots f(k_0))\right)\right), & t \in \mathbb{T}, \\ k_{t+1} = (1-e_t) f\left((1-e_{t-1}) f\left((1-e_{t-2}) f(\ldots f(k_0))\right)\right), & t \in \mathbb{T}. \end{cases} \tag{5}$$

[9] We apply the change of control variable similar to that of Phelps and Pollak (1968), and Peleg and Yaari (1973). They used as control variable savings rate, which in the Ramsey model is naturally related to consumption rate: $k_{t+1}/f(k_t) = s_t = 1-e_t$. However, in the decision-making context it seems reasonable to use consumption rate instead of savings rate.

Clearly, given k_0, there is a one-to-one correspondence between feasible consumption paths $\{c_t\}_{t\in\mathbb{T}}$ and feasible sequences of consumption rates $\{e_t\}_{t\in\mathbb{T}}$.

Let us incorporate the resource constraints into the objective function and, using (5), rewrite the utility function of agent i *in terms of consumption rates* as follows:

$$u_i\left(e_0 f(k_0)\right) + \delta_i u_i\left(e_1 f\left((1-e_0)f(k_0)\right)\right) + \delta_i^2 u_i\left(e_2 f\left((1-e_1)f\left((1-e_0)f(k_0)\right)\right)\right) + \ldots .$$

Then problem (3) becomes

$$\max \sum_{t\in\mathbb{T}} \delta_i^t u_i\left(e_t f\left((1-e_{t-1})f\left((1-e_{t-2})f(\ldots f(k_0))\right)\right)\right), \quad \text{s. t.} \quad 0 \le e_t \le 1,\ t\in\mathbb{T}. \quad (6)$$

Definition. *The solution to problem (6), $E^{i*} = \{e_t^{i*}\}_{t\in\mathbb{T}}$, is called an optimum in terms of consumption rates for agent i.*

It is clear that, given k_0, there is a one-to-one correspondence between optima in terms of consumption levels and optima in terms of consumption rates. It follows that there is a unique optimum in terms of consumption rates for each agent.

4.2 Intertemporal voting equilibria

We give the definition of intertemporal voting equilibrium in two stages, following the Hicks–Grandmont approach (Hicks, 1939; Grandmont, 1977). First, for an arbitrary point in time τ we define a time τ temporary voting equilibrium as a Condorcet winner in voting over the current consumption rate given a current stock of capital $k_\tau > 0$ and some expectations about future consumption rates. Secondly, we define an intertemporal voting equilibrium as a sequence, each element of which is a time τ (temporary) voting equilibrium provided agents have perfect foresight about future consumption rates.

Consider an arbitrary point in time τ. Suppose that the capital stock is $k_\tau > 0$ and the agents have some expectations about future consumption rates represented by a sequence $E_{\tau+1,T} = \{e_t\}_{t=\tau+1}^{T}$.[10] Preferences of agent i in voting over the time τ consumption rate are given by the following objective function:

$$V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T}) = \sum_{t=\tau}^{T} \delta_i^{t-\tau} u_i\left(e_t f\left((1-e_{t-1})f\left((1-e_{t-2})f(\ldots f(k_\tau))\right)\right)\right).$$

It is clear that the objective function is well-defined only if $V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T}) \neq \pm\infty$. In order to ensure that the objective function is finite, it is necessary to impose certain restrictions on the sequence of expectations $E_{\tau+1,T}$. We shall require that the sequence of expectations is ***non-degenerate***. The formal definition of a non-degenerate sequence can be found in Appendix A. Here it is sufficient to say that if the sequence of expectations, $E_{\tau+1,T}$, is non-degenerate, then for any $k_\tau > 0$, the function $V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})$ is well-defined and differentiable with respect to e_τ on the interval $(0,1)$.[11]

[10] For simplicity of presentation, we assume that all agents have the same expectations, though as a general rule each agent can have her own expectations.

[11] In particular, in the case of a finite horizon, the sequence $E_{\tau+1,T} = \{e_t\}_{t=\tau+1}^{T}$ is called non-degenerate if $0 < e_t < 1,\ t = \tau+1, t = \tau+2, ..., T-1,\ 0 < e_T \le 1$. In the case of an infinite horizon, the definition is a little more complicated.

Definition. *Given the stock of capital at time $\tau < T$, $k_\tau > 0$, and non-degenerate expectations $E_{\tau+1,T}$, we call e^i_τ a preferred time τ consumption rate for agent i if it is a solution to the one-dimensional optimization problem*

$$\max_{0 \le e_\tau \le 1} V^i_\tau(k_\tau, e_\tau, E_{\tau+1,T}).$$

If $T < \infty$, the preferred time T consumption rate for agent i is $e^i_T = 1$.

It should be emphasized that a preferred time τ consumption rate depends on the current capital stock and expectations.

Definition. *Given the current stock of capital at time $\tau < T$, $k_\tau > 0$, and non-degenerate expectations $E_{\tau+1,T}$, we call e^*_τ a time τ (temporary) voting equilibrium if it is a Condorcet winner in one-dimensional voting over the time τ consumption rate. If $T < \infty$, the time T voting equilibrium is $e^*_T = 1$.*

Now let us explain what we mean by an intertemporal voting equilibrium. Suppose that agents vote step by step starting from time 0. They are given the initial capital stock k_0 and some non-degenerate expectations about future consumption rates, $E_{1,T}$. The winner in voting over the time 0 consumption rate, the time 0 voting equilibrium e^*_0, generically depends on expectations. At time 1, all relevant information about the decision made at time 0 is gathered in the new capital stock k_1. Agents vote over the time 1 consumption rate given k_1 and some non-degenerate expectations about future consumption rates, $E_{2,T}$. And so on. If the agents have perfect foresight, then an outcome of this dynamic procedure is called an intertemporal voting equilibrium.

To give a formal definition, suppose that we are given an initial stock of capital, k_0 and a sequence of consumption rates, $E_{0,T} = \{e_\tau\}_{\tau=0}^T$. At every date τ, the current capital stock k_τ is unambiguously determined by the past consumption rates. Hence for every $\tau \in \mathbb{T}$, $k_{\tau+1}$ can be considered as a function of k_0 and the past values of consumption rates $E_{0,\tau} = \{e_0, e_1, \ldots, e_\tau\}$. For a given $k_0 > 0$, let us denote $k_{0,0} = k_0$, and recursively define the functions $k_{0,\tau}(\cdot,\cdot)$ as

$$k_{0,1}(k_0, E_{0,0}) = (1 - e_0) f\left(k_{0,0}\right),$$
$$k_{0,\tau}(k_0, E_{0,\tau-1}) = (1 - e_{\tau-1}) f\left(k_{0,\tau-1}(k_0, E_{0,\tau-2})\right), \quad \tau = 2, 3, \ldots.$$

Definition. *We call a non-degenerate sequence of consumption rates $E^*_{0,T} = \{e^*_\tau\}_{\tau=0}^T$ an intertemporal voting equilibrium starting from $k_0 > 0$ if for each τ, e^*_τ is a time τ voting equilibrium for the current stock of capital given by $k_\tau = k_{0,\tau}(k_0, E^*_{0,\tau-1})$[12] under perfect foresight about future outcomes of voting (i.e., under expectations given by $E^*_{\tau+1,T} = \{e^*_t\}_{t=\tau+1}^T$).*

Technically, an intertemporal voting equilibrium is a non-degenerate sequence, every element of which is chosen by a majority of agents provided all other consumption rates are already chosen according to the same procedure. Hence an intertemporal voting equilibrium is essentially a Kramer–Shepsle equilibrium.

[12] The sequence $E^*_{0,\tau-1}$ contains the first τ elements from an intertemporal voting equilibrium $E^*_{0,T}$, i.e., $E^*_{0,\tau-1} = \{e^*_t\}_{t=0}^{\tau-1}$.

5 Step-by-step intertemporal optimum

Before presenting our main results about an intertemporal voting equilibrium, let us gain more insight into the proposed voting procedure by analyzing the optimization problem in terms of consumption rates for a single agent. In this section we introduce the notion of a step-by-step intertemporal optimum, to which the notion of an intertemporal voting equilibrium is reduced in the case with only one agent, and show that it coincides with the optimum in terms of consumption rates. This simple result, which will be useful in what follows, seems not surprising and is in fact quite natural. However, to the best of our knowledge, it has not yet been explicitly stated and proved.

Consider problem (6) for an arbitrary agent, and omit the index i for the simplicity of notation:

$$\max \sum_{t\in\mathbb{T}} \delta^t u\left(e_t f\left((1-e_{t-1}) f\left((1-e_{t-2}) f(\dots f(k_0))\right)\right)\right), \quad \text{s. t.} \quad 0 \le e_t \le 1, \quad t \in \mathbb{T}. \quad (7)$$

Suppose that instead of solving this problem "once-and-for-all" at time 0, the agent tries to solve it in a step-by-step manner. Namely, suppose that at each time τ she determines e_τ by solving the problem

$$\max_{0\le e_\tau \le 1} V_\tau(k_\tau, e_\tau, E_{\tau+1,T}), \quad (8)$$

where

$$V_\tau(k_\tau, e_\tau, E_{\tau+1,T}) = \sum_{t=\tau}^{T} \delta^{t-\tau} u\left(e_t f\left((1-e_{t-1}) f\left((1-e_{t-2}) f(\dots f(k_\tau))\right)\right)\right), \quad (9)$$

$k_\tau > 0$ is the capital stock at time τ (it is determined in the previous step) and $E_{\tau+1,T} = \{e_t\}_{t=\tau+1}^T$ are expectations about future consumption rates. Clearly, the outcome of this procedure depends on the way expectations are formed. We are interested in the case where the agent has perfect foresight about her future decisions about consumption rate.

Definition. *Consider problem (7). We call a non-degenerate sequence of consumption rates $E^*_{0,T} = \{e^*_\tau\}_{\tau=0}^T$ a step-by-step intertemporal optimum if for each $\tau \in \mathbb{T}$, e^*_τ is a solution to the following problem:*

$$\max_{0\le e_\tau \le 1} V_\tau(k_\tau, e_\tau, E^*_{\tau+1,T}),$$

where $k_\tau = k_{0,\tau}(k_0, E^*_{0,\tau-1})$ and $E^*_{\tau+1,T} = \{e^*_t\}_{t=\tau+1}^T$.

If we call a solution to problem (8) *a time τ temporary optimum*, then a step-by-step intertemporal optimum is a sequence of temporary optima obtained under perfect foresight about future extractions rates. It should be noted that in an intertemporal voting equilibrium the agent has perfect foresight about future results of voting, while in step-by-step intertemporal optimum she has perfect foresight about her personal decisions.

Also it is noteworthy that if we adopt an atemporal point of view, then a step-by-step intertemporal optimum can simply be considered as a result of coordinate-wise maximization of the function $\sum_{t\in\mathbb{T}} \delta^t u\left(e_t f\left((1-e_{t-1}) f\left((1-e_{t-2}) f(\dots f(k_0))\right)\right)\right)$.

Clearly, if some sequence is an a "once-and-for-all" solution to problem (8), then this sequence is a step-by-step intertemporal optimum. There arises a question, whether the opposite is also true. Is it correct that any step-by-step intertemporal optimum is the *optimum in terms of consumption rates*?

The answer to this question is positive if either $T < \infty$ or $T = \infty$ and the felicity function $u(c)$ satisfies

Regularity condition. *There exists* $\gamma > 0$ *such that* $\lim_{c \to 0} c^{\gamma} u'(c) = 0$.

This condition means that $u'(c)$ tends to infinity at $c \to 0$ no faster than any power function. For instance, every CIES felicity function meets the regularity condition.[13] Moreover, every felicity function such that $u(0) > -\infty$ also satisfies this condition.

Proposition 1. *Consider problem (7). Suppose that* $T < \infty$*, or* $T = \infty$ *and the felicity function satisfies the regularity condition. A step-by-step intertemporal optimum exists and coincides with the optimum in terms of consumption rates.*

Proof. See Appendix B. □

It follows that there exists a unique (non-degenerate) step-by-step intertemporal optimum for each agent. It coincides with the unique *optimum in terms of consumption rates*, and corresponds to the unique *optimum in terms of consumption levels* for this agent.

6 Main Results

Now we are ready to characterize temporary and intertemporal voting equilibria. We begin by explicitly stating the existence of a time τ voting equilibrium.

Proposition 2. *For all non-degenerate expectations, the preferences of each agent in voting over the time* τ *consumption rates are strictly concave and single-peaked; hence the median voter theorem applies and a time* τ *voting equilibrium exists.*

Proof. See Appendix C. □

It follows that at each given point in time there exists an instantaneous Condorcet winner, which is by definition a time τ voting equilibrium. Proposition 2 states that our institutional framework (intertemporal majority voting) is well-defined even if expectations are not correct and differ for different agents. In any case, voting over the current consumption rate is a well-defined one-dimensional decision problem.

Now we can provide a characterization of an intertemporal voting equilibrium in the special but important case where all agents have the same felicity function, so that agents are heterogeneous only in their time preferences. First we characterize a time τ voting equilibrium.

Proposition 3. *Suppose all agents have the same felicity function and the same non-degenerate expectations. A time* τ *voting equilibrium exists, is unique, and coincides with the preferred time* τ *consumption rate for the agent with the median discount factor* δ_{med}.

Proof. Here we give only a sketch the proof. For the details, see Appendix C. We prove that if agents have the same felicity function and the same non-degenerate expectations, then higher values of the discount factor correspond to lower values of the preferred time τ consumption rate. Therefore it follows from Proposition 2 that a time τ voting equilibrium is the preferred consumption rate for the agent with the median discount factor. □

Combining Propositions 1 and 3, we can formulate the following Theorem.

[13] The class of felicity functions that satisfy the regularity condition is similar to that considered by Ekeland and Scheinkman (1986). In Case 2 (linear production function), this condition is redundant.

Theorem 1. *Suppose all agents have the same felicity function. Suppose further that $T < \infty$, or $T = \infty$ and the felicity function satisfies the regularity condition. An intertemporal voting equilibrium starting from any $k_0 > 0$ exists, is unique and coincides with the optimum in terms of consumption rates for the agent with the median discount factor δ_{med}.*

Proof. Proposition 3 states that the time τ voting equilibrium is the time τ temporary optimum for the agent with the median discount factor. Taking into account the definition of a step-by-step intertemporal optimum, it follows from Proposition 1 that an intertemporal voting equilibrium is the *optimum in terms of consumption rates* for the agent with the median discount factor. □

Since there is a one-to-one correspondence between the optimum in terms of consumption rates and the optimum in terms of consumption levels, *there is a one-to-one correspondence between the intertemporal voting equilibrium and the optimum in terms of consumption levels for the agent with the median discount factor*. Thus in the case where all agents have the same felicity function, the proposed voting procedure yields as the outcome the *optimum in terms of consumption levels* for the agent with the median discount factor.

Note that when $T = \infty$ in Case 2 (linear production function) the *optimum in terms of consumption levels* is a balanced-growth path. If all agents have the same CIES felicity function with the parameter ρ, then the solution to problem (3) is

$$c_{t+1}^{i*} = (\delta_i A)^{\frac{1}{\rho}} c_t^{i*}, \quad k_{t+1}^{i*} = (\delta_i A)^{\frac{1}{\rho}} k_t^{i*}, \quad t = 0, 1, \ldots.$$

Therefore, regardless of initial conditions, consumption and the capital stock for agent i grow at a constant rate $(\delta_i A)^{\frac{1}{\rho}}$. The corresponding intertemporal voting equilibrium is a constant sequence of consumption rates, $\{e^*, e^*, \ldots\}$, where e^* is determined by the median discount factor:

$$e^* = 1 - \frac{1}{A}\left(1 + (\delta_{med} A)^{\frac{1}{\rho}}\right).$$

7 Steady-state and balanced-growth voting equilibria

Now suppose that agents differ both in their time preferences and felicity functions. Clearly, there is no reason to expect that the result of Theorem 1 still holds in this case. Because of multi-dimensional heterogeneity, it is in principle impossible to claim that an intertemporal voting equilibrium is determined by the discount factor alone. However, we are able to obtain some results even with multi-dimensional heterogeneity. In Case 1 (strictly concave production function) these results concern steady-state voting equilibria, and in Case 2 (linear production function) — balanced-growth voting equilibria. We show the important difference between the two cases: in Case 1 (strictly concave production function), the steady-state voting equilibrium is fully determined by the median discount factor, whereas in Case 2 (linear production function), the balanced-growth voting equilibrium depends not only on the agents' discount factors, but also on the agents' intertemporal elasticity of substitution.

7.1 Case 1 (strictly concave production function): Steady-state voting equilibrium

If $T = \infty$ and the sequence of consumption rates is constant over time, $E = \{e, e, \ldots\}$, then the considered model becomes the Solow model with a unique steady state. The capital stock in this steady state, $k(e)$, is the only positive solution to the following equation[14] in k:

$$k = (1 - e)f(k). \tag{10}$$

Definition. *Consider Case 1 (strictly concave production function). We call* e^* *a steady-state voting equilibrium if the sequence* $\{e^*, e^*, \ldots\}$ *is an intertemporal voting equilibrium starting from* $k_0 = k(e^*) > 0$.

Suppose first that all agents have the same felicity function that satisfies the regularity condition. It follows from Theorem 1 that for any k_0 there is a unique intertemporal voting equilibrium, which corresponds to the optimum for the agent with the median discount factor. Take as the initial capital stock the value k^* determined by the "modified golden rule" for the agent with the median discount factor δ_{med}: $\delta_{med} f'(k^*) = 1$.

The optimum for the agent with the median discount factor starting from $k_0 = k^*$ is her steady-state optimum. The corresponding optimum in terms of consumption rates is a constant sequence $E^* = \{e^*, e^*, \ldots\}$, and, by Theorem 1, is a unique intertemporal voting equilibrium. Clearly, k^* is the unique solution to equation (10) at $e = e^*$. Hence e^*, the optimal consumption rate of the agent with the median discount factor δ_{med}, is the unique stationary voting equilibrium.

Since k^* is given by the "modified golden rule" and e^* depends only on the median discount factor, the stationary voting equilibrium does not depend on the felicity function of agents. This observation leads to the following theorem which holds in the general case where agents have different felicity functions.

Theorem 2. *In Case 1 (strictly concave production function),*

$$e^* = 1 - \frac{k^*}{f(k^*)} \tag{11}$$

is the unique steady-state voting equilibrium.

Proof. See Appendix D. □

Thus, even in the case with different felicity functions there is a unique steady-state voting equilibrium. It is completely determined by the "modified golden rule" for the agent with the median discount factor and independent on felicity functions.

It is well-known that in a single-agent Ramsey model, the optimal capital stock converges to the modified golden rule path, which is fully determined by the discount factor of the agent, and is independent of her felicity function. In our model, any intertemporal voting equilibrium converges to the steady-state voting equilibrium if all agents have the same felicity function. Is the same result true in the case where agents have different felicity functions? This, as well as the proof of the existence of an intertemporal voting equilibrium in the general case, may be a topic of further research.

[14] It exists if $(1 - e)f'(0) > 1$.

7.2 Case 2 (linear production function): Balanced-growth voting equilibrium

As we noted above, in Case 2 (linear production function) with $T = \infty$ if all agents have the same CIES felicity function, then the *optimum in terms of consumption levels* for each agent is a balanced-growth path, and the intertemporal voting equilibrium is characterized by a constant consumption rate.

If agents have different CIES felicity functions, then the *optimum in terms of consumption levels* for each agent i is also a balanced-growth path in which consumption and capital grow at a constant rate γ_i given by

$$1 + \gamma_i = (\delta_i A)^{\frac{1}{\rho_i}} . \tag{12}$$

Though agents differ both in the discount factors and in the elasticities of intertemporal substitution, due to the linear production function, their heterogeneity can in some sense be considered as one-dimensional, because agents are naturally characterized by their growth rates that aggregate both heterogeneity parameters.

It is natural to conjecture that any intertemporal voting equilibrium in this case is also characterized by a constant growth rate. At the moment, we cannot prove this conjecture. However, we shall prove now that a *balanced-growth voting equilibrium* exists and, what is important, *is determined not by the median discount factor, but by the median growth rate* γ_{med}.

If we are given a constant over time consumption rate e, then for any initial stock k_0 the corresponding capital stock and consumption grow at a constant rate:

$$k_{t+1} = (1+\gamma)k_t, \quad c_{t+1} = (1+\gamma)c_t, \quad t = 0, 1, \dots,$$

where $\gamma = (1-e)A - 1$.

Definition. *Consider Case 2 (linear production function). We call e^* a balanced-growth voting equilibrium starting from $k_0 > 0$ if the sequence $\{e^*, e^*, \dots\}$ is an intertemporal voting equilibrium starting from k_0.*

The following theorem shows that a balanced-growth voting equilibrium is fully determined by the preferences of the agent with the median growth rate γ_{med}.

Theorem 3. *In Case 2 (linear production function), for any $k_0 > 0$, there is a unique balanced-growth voting equilibrium starting from $k_0 > 0$. It is given by*

$$e^* = 1 - \frac{1}{A}(1 + \gamma_{med}). \tag{13}$$

Proof. See Appendix E. □

Clearly, the consumption path corresponding to the balanced-growth voting equilibrium $E^* = \{e^*, e^*, \dots\}$ is the balanced-growth path for the agent with the median growth rate γ_{med}. However, it is a topic for further research whether there exist intertemporal voting equilibria that are not balanced-growth voting equilibria.

8 Conclusion

The problem of collective choice naturally arises in many economic applications with heterogeneous agents. In this paper, we study a Ramsey-type growth model with common consumption and agents who may differ in their instantaneous utility ("felicity") functions and discount factors. It is well known that, in general, there is no Condorcet winner if agents vote over feasible consumption streams. This holds even if agents differ only in their discount factors and heterogeneity is one-dimensional. Notwithstanding these negative findings, we show in this paper that the choice of the optimal consumption stream of the "median" agent can be obtained as the result of a simple and natural institutional setup, intertemporal majority voting, in many important cases.

Our voting procedure is based on two principles. First, agents vote step-by-step at each point in time. Second, agents vote over the consumption *rate*, not over the consumption *level*. We define a temporary voting equilibrium, which results in the Condorcet winner among all current consumption rates under some expectations about future consumption rates. Then, we define an intertemporal voting equilibrium as a sequence of temporary voting equilibria under the assumption that agents have perfect foresight about future consumption rates. From the technical point of view, an intertemporal voting equilibrium is a Kramer–Shepsle equilibrium in terms of consumption rates.

Our main result concerns the case where agents have identical felicity functions and differ only in their discount factors. We prove that an intertemporal voting equilibrium exists, is unique, and coincides with the optimum in terms of consumption rates for the agent with the median discount factor. We thus show that even in the absence of a Condorcet winner there is a stable outcome of intertemporal majority voting. Since this outcome is determined by the preferences of the agent with the median discount factor, it is both time-consistent and Pareto efficient.

We also consider the general framework where agents may differ also in their felicity functions, and analyze two special cases. In the case of a strictly concave production function and arbitrary felicity functions, we define a steady-state voting equilibrium, and show that it is unique and is fully determined by the median discount factor. Our analysis suggests that in the case of a strictly concave production function the analogy with the standard Ramsey model may fruitfully be applied. One may conjecture that every intertemporal voting equilibrium converges to the steady-state voting equilibrium, and thus the winner of the voting procedure eventually depends only on the discount factor. However, further research is needed to confirm or reject this conjecture.

In the case of a linear production function and CIES felicity functions, we define a balanced-growth voting equilibrium, prove its uniqueness, and show that it is determined by the median growth rate, where the growth rate for each agent depends not only on the discount factor, but also on the elasticity of intertemporal substitution.

Finally, it should be recognized that on the one hand, in our model agents are excessively sophisticated because in an intertemporal voting equilibrium they have perfect foresight about future outcomes of voting. On the other hand, they are sophisticated to a limited extent, because the set of their strategies is limited (consumption depends on production in a linear way). Introducing either less or more sophisticated agents into our framework might be a topic for further research.

A Non-degenerate sequences and properties of the objective functions

A.1 Useful notation

For an arbitrary τ and a sequence of consumption rates $E_{\tau,T} = \{e_\tau, e_{\tau+1}, \dots, e_T\}$, denote

$$E_{\tau,\tau} = \{e_\tau\}, \quad E_{\tau,t} = \{e_\tau, e_{\tau+1}, \dots, e_t\} = \{e_\tau, E_{\tau+1,t}\},\ t = \tau+1, \tau+2, \dots,$$

and

$$k_{\tau,\tau} = k_\tau, \quad k_{\tau,\tau+1}(k_\tau, E_{\tau,\tau}) = (1-e_\tau) f\left(k_{\tau,\tau}\right),$$
$$k_{\tau,t+1}(k_\tau, E_{\tau,t}) = (1-e_t) f\left(k_{\tau,t}(k_\tau, E_{\tau,t-1})\right),\ t = \tau+1, \tau+2, \dots.$$

Let also

$$f_{\tau,\tau} = f\left(k_\tau\right), \quad f_{\tau,\tau+1}(k_\tau, E_{\tau,\tau}) = f\left((1-e_\tau) f_{\tau,\tau}\right),$$
$$f_{\tau,t+1}(k_\tau, E_{\tau,t}) = f\left((1-e_t) f_{\tau,t}(k_\tau, E_{\tau,t-1})\right),\ t = \tau+1, \tau+2, \dots.$$

Thus for $t > \tau$,

$$k_{\tau,t+1}(k_\tau, E_{\tau,t}) = (1-e_t) f_{\tau,t}\left(k_\tau, E_{\tau,t-1}\right),\ f_{\tau,t+1}(k_\tau, E_{\tau,t}) = f\left(k_{\tau,t+1}(k_\tau, E_{\tau,t})\right).$$

For simplicity of notation, we often drop the arguments of these functions when they play no significant role. However, the reader should bear in mind that $f_{\tau,t+1}$ is a function of k_τ and the $t-\tau+1$ consumption rates $\{e_\tau, e_{\tau+1}, \dots, e_t\}$.

The derivatives of $f_{\tau,t+1}$ can be obtained using the chain rule of differentiation:

$$\frac{\partial f_{\tau,t+1}}{\partial e_t} = -f'(k_{\tau,t+1}) f_{\tau,t},\ \frac{\partial f_{\tau,t+1}}{\partial e_{t-1}} = -f'(k_{\tau,t+1})(1-e_t) f'(k_{\tau,t}) f_{\tau,t-1},$$
$$\frac{\partial f_{\tau,t+1}}{\partial e_{t-2}} = -f'(k_{\tau,t+1})(1-e_t) f'(k_{\tau,t})(1-e_{t-1}) f'(k_{\tau,t-1}) f_{\tau,t-2},\ \dots.$$

It is clear that the derivative of $f_{\tau,t+1}(k_\tau, E_{\tau,t})$ with respect to each consumption rate $\{e_\tau, e_{\tau+1}, \dots, e_t\}$ is negative.

A.2 Non-degenerate sequences of consumption rates

Definition. *A) Suppose that $T < \infty$. We call a sequence $E_{\tau,T} = \{e_t\}_{t=\tau}^{T}$ non-degenerate if*

$$0 < e_t < 1,\ t = \tau, \tau+1, \dots, T-1; \quad 0 < e_T \le 1.$$

B) Suppose that $T = \infty$ and consider a sequence $E_{\tau,\infty} = \{e_t\}_{t=\tau}^{\infty}$ such that $0 < e_t < 1$ for all t.

1. *In Case 1 (strictly concave production function), the sequence $E_{\tau,\infty} = \{e_t\}_{t=\tau}^{\infty}$ is called non-degenerate if*

   1.1.
   $$0 < \liminf_{t\to\infty} e_t \le \limsup_{t\to\infty} e_t < 1, \tag{14}$$

   1.2. for some $\tilde{k}_\tau > 0$ the sequence $\{\tilde{k}_t\}_{t=\tau}^{\infty}$ given by
   $$\tilde{k}_{t+1} = (1-e_t) f(\tilde{k}_t), \quad t = \tau, \tau+1, \dots,$$
   satisfies
   $$\liminf_{t\to\infty} \tilde{k}_t > 0. \tag{15}$$

2. *In Case 2 (linear production function), the sequence $E_{\tau,\infty} = \{e_t\}_{t=\tau}^{\infty}$ is called non-degenerate if there exist $\underline{e}$ and $\bar{e}$ such that*

2.1.

$$0 \le \underline{e} < \liminf_{t\to\infty} e_t \le \limsup_{t\to\infty} e_t < \bar{e} \le 1, \tag{16}$$

2.2. for all i,[15]

$$\begin{cases} \delta_i \left(A(1-\underline{e})\right)^{1-\rho_i} < 1, & \text{if} \quad \rho_i \le 1, \\ \delta_i \left(A(1-\bar{e})\right)^{1-\rho_i} < 1, & \text{if} \quad \rho_i > 1. \end{cases} \tag{17}$$

A.3 Properties of the objective functions

We need the above definition to establish certain important properties of the agents' objective functions. Recall that the objective function of agent i in voting over the time τ consumption rate is given by

$$V^i_\tau(k_\tau, e_\tau, E_{\tau+1,T}) = u_i\left(e_\tau f_{\tau,\tau}\right) + \delta_i u_i\left(e_{\tau+1} f_{\tau,\tau+1}\right) + \delta_i^2 u_i\left(e_{\tau+2} f_{\tau,\tau+2}\right) + \dots.$$

Let us show that if the sequence of expectations $E_{\tau+1,T}$ is non-degenerate, then for any $k_\tau > 0$ the objective function of agent i is well-defined: $-\infty < V^i_\tau(k_\tau, e_\tau, E_{\tau+1,T}) < +\infty$.

When $T < \infty$, the finiteness of $V^i_\tau(k_\tau, e_\tau, E_{\tau+1,T})$ is evident, since the expected consumption rates lie strictly between 0 and 1 (except for the time T).

Suppose $T = \infty$ and consider Case 1 (strictly concave production function). Since there is a maximum sustainable stock, $\bar{k} = f(\bar{k})$ (see (1)), condition (14) guarantees that $V^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty})$ is bounded from above. Conditions (14) and (15)[16] also ensure that for any $k_\tau > 0$ the path of the capital stock $\{k_t\}_{t=\tau}^\infty$ constructed by $k_{t+1} = (1-e_t)f(k_t)$ and the corresponding consumption path $\{c_t\}_{t=\tau}^\infty$ constructed by $c_t = e_t f(k_t)$ are separated from zero:

$$\liminf_{t\to\infty} k_t > 0, \quad \liminf_{t\to\infty} f(k_t) > 0, \quad \liminf_{t\to\infty} c_t > 0. \tag{18}$$

Indeed, if $k_\tau > \tilde{k}_\tau$, then $k_t > \tilde{k}_t$, $f(k_t) > f(\tilde{k}_t)$ and $c_t > \tilde{c}_t$ for all $t = \tau, \tau+1, \dots$. If $k_\tau < \tilde{k}_\tau$, then for all $t = \tau, \tau+1, \dots$, $k_t < \tilde{k}_t$ and $k_{t+1}/\tilde{k}_{t+1} > k_t/\tilde{k}_t$, because $f(k)/k$ is decreasing in k. Therefore $\lim_{t\to\infty} k_t/\tilde{k}_t > 0$, $\lim_{t\to\infty} f(k_t)/f(\tilde{k}_t) > 0$ and $\lim_{t\to\infty} c_t/\tilde{c}_t > 0$. Hence, in both cases the inequalities in (18) hold true and $V^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty})$ is also bounded from below.

Consider Case 2 (linear production function). The objective function of agent i takes the form[17]

$$V^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty}) = \sum_{t=\tau}^{\infty} \frac{\delta_i^{t-\tau}}{1-\rho_i} \left(A^{t-\tau+1} e_t (1-e_{t-1})(1-e_{t-2})\cdots(1-e_\tau)k_\tau\right)^{1-\rho_i}.$$

For i such that $0 < \rho_i < 1$ it follows from (16) and (17) that

$$\begin{aligned} 0 < V^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty}) &\le \frac{1}{1-\rho_i}\left(Ae_\tau k_\tau\right)^{1-\rho_i} + \frac{1}{1-\rho_i}\delta_i\left(A^2(1-e_\tau)k_\tau\right)^{1-\rho_i} \cdot \\ &\cdot \left[(\bar{e})^{1-\rho_i} + \delta_i A^{1-\rho_i}(\bar{e}(1-\underline{e}))^{1-\rho_i} + (\delta_i A^{1-\rho_i})^2(\bar{e}(1-\underline{e})^2)^{1-\rho_i} + \dots\right] \\ &= \frac{1}{1-\rho_i}\left(Ae_\tau k_\tau\right)^{1-\rho_i} + \frac{1}{1-\rho_i}\delta_i\left(A^2(1-e_\tau)k_\tau\right)^{1-\rho_i}(\bar{e})^{1-\rho_i} \cdot \\ &\cdot \left[1 + \delta_i(A(1-\underline{e}))^{1-\rho_i} + (\delta_i(A(1-\underline{e}))^{1-\rho_i})^2 + \dots\right] \\ &= \frac{1}{1-\rho_i}\left(Ae_\tau k_\tau\right)^{1-\rho_i} + \frac{\delta_i}{1-\rho_i}\frac{\left(A^2(1-e_\tau)k_\tau\right)^{1-\rho_i}(\bar{e})^{1-\rho_i}}{1-\delta_i\left(A(1-\underline{e})\right)^{1-\rho_i}} < +\infty. \end{aligned}$$

Slightly modifying the above argument, for i such that $\rho_i > 1$ we obtain that

$$0 > V^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty}) > \frac{1}{1-\rho_i}\left(Ae_\tau k_\tau\right)^{1-\rho_i} + \frac{\delta_i}{1-\rho_i}\frac{\left(A^2(1-e_\tau)k_\tau\right)^{1-\rho_i}(\underline{e})^{1-\rho_i}}{1-\delta_i\left(A(1-\bar{e})\right)^{1-\rho_i}} > -\infty.$$

[15] Recall that in Case 2 (linear production function) the felicity function is given by (2).

[16] Note that if $f'(0) = +\infty$, then condition (15) is redundant, because it follows from condition (14).

[17] Here and hereafter we consider $\rho_i \ne 1$. The case with the logarithmic felicity function ($\rho_i = 1$) can be considered similarly.

Thus we have shown that for a non-degenerate sequence of expectations $E_{\tau+1,T}$, the objective function of agent i exists and is well-defined. However, we also need to prove that $V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})$ is continuously differentiable with respect to e_τ. By differentiating the objective function term by term, we get

$$\frac{\partial V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})}{\partial e_\tau} = f(k_\tau)\left(\Phi_\tau^i(k_\tau, e_\tau) - \Psi_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})\right), \tag{19}$$

where

$$\Phi_\tau^i(k_\tau, e_\tau) = u_i'\left(e_\tau f(k_\tau)\right), \tag{20}$$

and

$$\begin{aligned} \Psi_\tau^i(k_\tau, e_\tau, E_{\tau+1,T}) &= \delta_i u_i'\left(e_{\tau+1} f_{\tau,\tau+1}\right) e_{\tau+1} f'\left((1-e_\tau) f_{\tau,\tau}\right) \\ &+ \delta_i^2 u_i'\left(e_{\tau+2} f_{\tau,\tau+2}\right) e_{\tau+2} f'\left((1-e_{\tau+1}) f_{\tau,\tau+1}\right)(1-e_{\tau+1}) f'\left((1-e_\tau) f_{\tau,\tau}\right) + \dots \\ &+ \delta_i^s u_i'\left(e_{\tau+s} f_{\tau,\tau+s}\right) e_{\tau+s}\left[\prod_{k=1}^{s-1}\left[f'\left((1-e_{\tau+k}) f_{\tau,\tau+k}\right)(1-e_{\tau+k})\right]\right] f'\left((1-e_\tau) f_{\tau,\tau}\right) + \dots . \end{aligned} \tag{21}$$

The following Lemma shows that term-by-term differentiation is valid.

Lemma 1. *Suppose that the sequence of expectations $E_{\tau+1,T}$ is non-degenerate. For any $k_\tau > 0$, $V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})$ is continuously differentiable with respect to e_τ on the interval $(0, 1)$, and its derivative is given by (19).*

Proof. When $T < \infty$, the statement of the lemma is evident. When $T = \infty$, the proof is based on a well-known theorem of analysis (see, e.g., Zorich, 2015, p. 388). We need to show that $\Psi_\tau^i(k_\tau, e_\tau, E_{\tau+1,\infty})$, which is the infinite series of continuous functions, is uniformly convergent on the interval $\{e_\tau \mid \xi \le e_\tau \le 1-\xi\}$ for any $0 < \xi < 1$.

Consider Case 1 (strictly concave production function). It follows from (14) and (18) that

$$0 < \liminf_{t\to\infty} u'\left(e_{\tau+t} f_{\tau,\tau+t}\right) e_{\tau+t} \le \limsup_{t\to\infty} u'\left(e_{\tau+t} f_{\tau,\tau+t}\right) e_{\tau+t} < +\infty.$$

Using the fact that $f'(k) \le f(k)/k$ for any k, for $s = 2, 3, \dots$ we have

$$\begin{aligned} \prod_{t=1}^{s-1}\left[f'\left((1-e_{\tau+t}) f_{\tau,\tau+t}\right)(1-e_{\tau+t})\right] &\le \prod_{t=1}^{s-1}\left[\frac{f\left((1-e_{\tau+t}) f_{\tau,\tau+t}\right)}{(1-e_{\tau+t}) f_{\tau,\tau+t}}(1-e_{\tau+t})\right] \\ &= \prod_{t=1}^{s-1}\left[\frac{f_{\tau,\tau+t+1}}{f_{\tau,\tau+t}}\right] = \frac{f_{\tau,\tau+s}}{f_{\tau,\tau+1}} = \frac{f_{\tau,\tau+s}}{f\left((1-e_\tau) f_{\tau,\tau}\right)} \le \frac{\bar f}{f\left((1-e_\tau) f_{\tau,\tau}\right)} \le \frac{\bar f}{f\left(\xi f_{\tau,\tau}\right)}, \end{aligned}$$

where $\bar f = \max\{f(\bar k), f(k_\tau)\}$. The penultimate inequality follows from the existence of a maximum sustainable stock $\bar k$, and from the fact that if $k_t > \bar k$, then $k_{t+1} < k_t$.

By the Weierstrass M-test (see, e.g., Zorich, 2015, p. 374), $\Psi_\tau^i(k_\tau, e_\tau, E_{\tau+1,\infty})$ converges uniformly on the interval $\{e_\tau \mid \xi \le e_\tau \le 1-\xi\}$ for any $0 < \xi < 1$, and hence, by the uniform limit theorem (see, e.g., Zorich, 2015, p. 383), is continuous in e_τ. It follows that for a non-degenerate sequence $E_{\tau+1,\infty}$, $\frac{\partial V_\tau^i(k_\tau, e_\tau, E_{\tau+1,\infty})}{\partial e_\tau}$ exists, is continuous in e_τ, and is given by (19).

Consider now Case 2 (linear production function). The function $\Psi_\tau^i(k_\tau, e_\tau, E_{\tau+1,\infty})$ defined by (21) takes the form

$$\begin{aligned} \Psi_\tau^i(k_\tau, e_\tau, E_{\tau+1,\infty}) &= \delta_i A e_{\tau+1}\left(e_{\tau+1}(1-e_\tau) A^2 k_\tau\right)^{-\rho_i} \\ &+ \delta_i^2 e_{\tau+2}(1-e_{\tau+1}) A^2\left(e_{\tau+2}(1-e_{\tau+1})(1-e_\tau) A^3 k_\tau\right)^{-\rho_i} + \dots = \delta_i A (A^2 k_\tau)^{-\rho_i}(1-e_\tau)^{-\rho_i} \cdot \\ &\cdot \left[(e_{\tau+1})^{1-\rho_i} + \delta_i A^{1-\rho_i}\left(e_{\tau+2}(1-e_{\tau+1})\right)^{1-\rho_i} + \left(\delta_i A^{1-\rho_i}\right)^2\left(e_{\tau+3}(1-e_{\tau+2})(1-e_{\tau+1})\right)^{1-\rho_i} + \dots\right]. \end{aligned}$$

Applying the same argument as in the proof of the finiteness of the objective function in Case 2, we obtain that for i such that $0 < \rho_i < 1$ it follows from (16) and (17) that

$$0 < \Psi_\tau^i(k_\tau, e_\tau, E_{\tau+1,\infty}) < \frac{\delta_i A (A^2 k_\tau)^{-\rho_i}\left(\bar e\right)^{1-\rho_i}(1-e_\tau)^{-\rho_i}}{1-\delta_i\left(A(1-\underline{e})\right)^{1-\rho_i}} < \frac{\delta_i A (A^2 k_\tau)^{-\rho_i}\left(\bar e\right)^{1-\rho_i}\xi^{-\rho_i}}{1-\delta_i\left(A(1-\underline{e})\right)^{1-\rho_i}}.$$

Analogously, it can be shown that for i such that $\rho_i > 1$,

$$0 < \Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty}) < \frac{\delta_i A(A^2 k_\tau)^{-\rho_i} \left(\underline{e}\right)^{1-\rho_i} (1-e_\tau)^{-\rho_i}}{1-\delta_i \left(A(1-\bar{e})\right)^{1-\rho_i}} < \frac{\delta_i A(A^2 k_\tau)^{-\rho_i} \left(\underline{e}\right)^{1-\rho_i} \xi^{-\rho_i}}{1-\delta_i \left(A(1-\bar{e})\right)^{1-\rho_i}}$$

It follows that $\Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty})$ converges uniformly on the interval $\{e_\tau \mid \xi \le e_\tau \le 1-\xi\}$ for any $0 < \xi < 1$, and hence is continuous in e_τ. Therefore, for a non-degenerate sequence $E_{\tau+1,\infty}$, $\frac{\partial V^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty})}{\partial e_\tau}$ exists, is continuous in e_τ, and is given by (19). □

Remark 1. *Note that $\Phi^i_\tau(k_\tau, 0) = +\infty$, $\Phi^i_\tau(k_\tau, e_\tau)$ is continuous and strictly decreasing in e_τ for any $0 < e_\tau \le 1$. Furthermore, $\Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,T})$ is continuous in e_τ, and, due to the strict concavity of u_i and f, is strictly increasing in e_τ. Each term in $\Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,T})$ contains a multiplier of the form*

$$u'_i(e_t f_{\tau,t}) = u'_i\left(e_t f\left((1-e_{t-1}) f\left((1-e_{t-2}) f\left(\cdots f\left((1-e_\tau) f(k_\tau)\right)\right)\right)\right)\right).$$

If $e_\tau = 1$, due to the fact that $f(0) = 0$, $\Psi^i_\tau(k_\tau, 1, E_{\tau+1,T}) = +\infty$. Taking into account Lemma 1, we obtain that there exists a unique interior solution to the equation $\Phi^i_\tau(k_\tau, e_\tau) = \Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,T})$ in e_τ.

B Proof of Proposition 1

B.1 First-order conditions

Consider problem (3) with omitted index i:

$$\max \sum_{t\in\mathbb{T}} \delta^t u(c_t), \quad \text{s. t.} \quad c_t + k_{t+1} = f(k_t), \quad c_t \ge 0, \quad k_{t+1} \ge 0, \quad t \in \mathbb{T}.$$

The first-order conditions are given by

$$\frac{u'(c_t)}{u'(c_{t+1})} = \delta f'(k_{t+1}), \quad t < T. \tag{22}$$

The transversality condition in the case of a finite horizon is as follows:

$$k_{T+1} = 0. \tag{23}$$

In the infinite horizon case, the transversality condition is given by

$$\lim_{t\to\infty} \delta^t u'(c_t) k_{t+1} = 0. \tag{24}$$

Let us show that the sequence $\{c_t, k_{t+1}\}_{t=0}^\infty$ corresponding to a step-by-step intertemporal optimum, satisfies the first-order conditions (22).

Let τ be an arbitrary point in time, $k_\tau > 0$, and $E_{\tau,T}$ be a non-degenerate sequence of consumption rates. Consider the objective function $V_\tau(k_\tau, e_\tau, E_{\tau+1,T})$ given by (9). By Remark 1, a step-by-step intertemporal optimum is a solution to the following system of equations[18]:

$$\frac{\partial V_t\left(k_{0,t}(k_0, E_{0,t-1}), e_t, E_{t+1,T}\right)}{\partial e_t} = 0, \quad t < T. \tag{25}$$

Consider two adjacent equations of the system (25), for $t = \tau$ and for $t = \tau + 1$. It follows from (19)–(21) (again omitting index i) that the equation for $t = \tau$ is as follows:

$$\begin{aligned} u'(e_\tau f_{0,\tau}) = {} & \delta e_{\tau+1} u'(e_{\tau+1} f_{0,\tau+1}) f'(k_{0,\tau+1}) + \delta^2 (1-e_{\tau+1}) e_{\tau+2} u'(e_{\tau+2} f_{0,\tau+2}) f'(k_{0,\tau+2}) f'(k_{0,\tau+1}) \\ & + \delta^3 (1-e_{\tau+1})(1-e_{\tau+2}) e_{\tau+3} u'(e_{\tau+3} f_{0,\tau+3}) f'(k_{0,\tau+3}) f'(k_{0,\tau+2}) f'(k_{0,\tau+1}) + \ldots . \end{aligned} \tag{26}$$

Note that the right-hand side of the above equation can be rewritten as

$$\begin{aligned} & \delta f'(k_{0,\tau+1}) e_{\tau+1} u'(e_{\tau+1} f_{0,\tau+1}) + \delta f'(k_{0,\tau+1})(1-e_{\tau+1}) \cdot \\ & \quad \cdot \left[\delta e_{\tau+2} u'(e_{\tau+2} f_{0,\tau+2}) f'(k_{0,\tau+2}) + \delta^2 (1-e_{\tau+2}) e_{\tau+3} u'(e_{\tau+3} f_{0,\tau+3}) f'(k_{0,\tau+3}) f'(k_{0,\tau+2}) + \ldots\right]. \end{aligned} \tag{27}$$

[18] If $T < +\infty$ then there is an additional equation $e_T = 1$.

The equation for $t=\tau+1$ is as follows:

$$
\begin{aligned}
&u'(e_{\tau+1}f_{0,\tau+1})\\
&\quad= \delta e_{\tau+2}u'(e_{\tau+2}f_{0,\tau+2})f'(k_{0,\tau+2}) + \delta^2(1-e_{\tau+2})e_{\tau+3}u'(e_{\tau+3}f_{0,\tau+3})f'(k_{0,\tau+3})f'(k_{0,\tau+2}) + \dots.
\end{aligned}
$$

Substituting the right-hand side of the above equation into (27), we infer from (26) that

$$
\begin{aligned}
&u'(e_\tau f_{0,\tau})\\
&\quad= \delta f'(k_{0,\tau+1})\left(e_{\tau+1}u'(e_{\tau+1}f_{0,\tau+1}) + (1-e_{\tau+1})u'(e_{\tau+1}f_{0,\tau+1})\right) = \delta f'(k_{0,\tau+1})u'(e_{\tau+1}f_{0,\tau+1}).
\end{aligned}
$$

Applying this argument for $0 \le \tau < T$, we obtain that the system of equations (25) is equivalent to the system

$$
u'(e_t f_{0,t}) = \delta f'(k_{0,t+1})u'(e_{t+1}f_{0,t+1}), \quad t<T. \tag{28}
$$

Now it is straightforward to see that the mapping defined by (5) converts the system of equations (28) to the system of the first-order conditions (22).

B.2 Transversality condition

It remains to show that the sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ corresponding to a step-by-step intertemporal optimum satisfies the transversality condition.

Consider the case $T<\infty$. Then the transversality condition (23) follows from the fact that the optimal consumption rate at time T for every agent is $e_T=1$. In the last period it is optimal to consume everything.

Consider the case $T=\infty$. Iterating equation for $t=\tau$ from (28), we can replace expressions of the form

$$
\delta^{t-(\tau+1)}u'(e_t f_{0,t})f'(k_{0,t})f'(k_{0,t-1})\cdot\ldots\cdot f'(k_{0,\tau+1})
$$

on the right-hand side of equation (26) with $u'(e_\tau f_{0,\tau})$. Hence equation (26) can be rewritten as follows:

$$
\begin{aligned}
&u'(e_\tau f_{0,\tau}) - e_{\tau+1}u'(e_\tau f_{0,\tau}) - (1-e_{\tau+1})e_{\tau+2}u'(e_\tau f_{0,\tau})\\
&- (1-e_{\tau+1})(1-e_{\tau+2})e_{\tau+3}u'(e_\tau f_{0,\tau}) - (1-e_{\tau+1})(1-e_{\tau+2})(1-e_{\tau+3})e_{\tau+4}u'(e_\tau f_{0,\tau}) - \ldots = 0.
\end{aligned} \tag{29}
$$

Regrouping the terms on the left-hand side of (29), we get

$$
\begin{aligned}
&u'(e_\tau f_{0,\tau})(1-e_{\tau+1}) - e_{\tau+2}(1-e_{\tau+1})u'(e_\tau f_{0,\tau})\\
&\quad - (1-e_{\tau+1})(1-e_{\tau+2})e_{\tau+3}u'(e_\tau f_{0,\tau}) - (1-e_{\tau+1})(1-e_{\tau+2})(1-e_{\tau+3})e_{\tau+4}u'(e_\tau f_{0,\tau}) - \ldots\\
&\quad = u'(e_\tau f_{0,\tau})(1-e_{\tau+1})(1-e_{\tau+2}) - (1-e_{\tau+1})(1-e_{\tau+2})e_{\tau+3}u'(e_\tau f_{0,\tau})\\
&\quad\quad - (1-e_{\tau+1})(1-e_{\tau+2})(1-e_{\tau+3})e_{\tau+4}u'(e_\tau f_{0,\tau}) - \ldots\\
&= u'(e_\tau f_{0,\tau})(1-e_{\tau+1})(1-e_{\tau+2})(1-e_{\tau+3}) - (1-e_{\tau+1})(1-e_{\tau+2})(1-e_{\tau+3})e_{\tau+4}u'(e_\tau f_{0,\tau}) - \dots.
\end{aligned}
$$

Repeating this argument, (29) finally can be rewritten as

$$
u'(e_\tau f_{0,\tau})(1-e_{\tau+1})(1-e_{\tau+2})(1-e_{\tau+3})(1-e_{\tau+4})\cdots = 0.
$$

Since τ is chosen arbitrarily, and $u'(e_\tau f_{0,\tau})>0$, it follows that $\prod_{t=1}^{\infty}(1-e_t)=0$, which is equivalent to

$$
\sum_{t=1}^{\infty} e_t = +\infty. \tag{30}
$$

Now let us show that the sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ corresponding to the solution to the system (25) satisfies the transversality condition (24).

B.2.1 Case 1 (strictly concave production function)

Let us analyze the possible dynamics of the sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$. We begin with two lemmas.

Lemma 2. *Suppose that $\delta f'(k_{\Theta+1})>1$ for some Θ, and $k_{\Theta+1}\le k_\Theta$. Then the transversality condition (24) holds.*

Proof. Let us show that the conditions of the lemma imply $k_{t+1} < k_t$ for all $t \geq \Theta$. Indeed, by (22), $u'(c_\Theta) > u'(c_{\Theta+1})$, or $c_{\Theta+1} > c_\Theta$. Hence, $k_{\Theta+2} - k_{\Theta+1} = (f(k_{\Theta+1}) - f(k_\Theta)) + (c_\Theta - c_{\Theta+1}) < 0$. Thus $k_{\Theta+2} < k_{\Theta+1}$ and hence $\delta f'(k_{\Theta+2}) > 1$. Repeating the argument, we infer that for all $t > \Theta$, $k_{t+1} < k_\Theta$, and $u'(c_t) < u'(c_\Theta)$. Therefore, starting from $t = \Theta$, $\delta^t u'(c_t)k_{t+1} < \delta^t u'(c_\Theta)k_\Theta$, which implies (24). □

Lemma 3. *Suppose that there exists Θ such that $\delta f'(k_{t+1}) \leq 1$ for all $t \geq \Theta$. Then the transversality condition (24) holds.*

Proof. Recall that there is $\bar{k}$ such that $0 < f(\bar{k}) = \bar{k} < +\infty$, and thus k_t is bounded from above. Since $\delta f'(0) > 1$, it follows that for $t > \Theta$, $k_t \geq (f')^{-1}(1/\delta) > 0$. Hence $f(k_t) \geq f((f')^{-1}(1/\delta)) > 0$, and from (4) we get

$$c_t = e_t f(k_t) \geq e_t f\left((f')^{-1}(1/\delta)\right), \quad t > \Theta.$$

Therefore, due to (30),

$$\sum_{t=\Theta}^{\infty} c_t = +\infty. \tag{31}$$

It follows from (22) that $u'(c_{t+1}) \geq u'(c_t)$, $t \geq \Theta$. Therefore, the sequence $\{c_t\}_{t=\Theta}^{\infty}$ is monotonically non-increasing and converges. Suppose that $c_t \to 0$ as $t \to \infty$. Then $k_t \to \bar{k} = f(\bar{k})$, and since $f'(\bar{k}) < 1$, for some $0 < \xi < 1$ there exists Θ' such that for all $t > \Theta'$, $\delta f'(k_{t+1}) < 1 - \xi$. Thus $u'(c_t) = \delta f'(k_{t+1})u'(c_{t+1}) < (1-\xi)u'(c_{t+1})$. At the same time, it follows from the regularity condition that for some $\gamma > 0$, the sequence $u'(c_t)c_t^\gamma$ converges, and hence there exists Θ'' such that for all $t > \Theta''$,

$$\frac{u'(c_{t+1})c_{t+1}^\gamma}{u'(c_t)c_t^\gamma} < 1 + \xi.$$

Thus for all $t > \max\{\Theta', \Theta''\}$,

$$\frac{c_{t+1}}{c_t} < \left((1+\xi)\frac{u'(c_t)}{u'(c_{t+1})}\right)^{\frac{1}{\gamma}} < ((1+\xi)(1-\xi))^{\frac{1}{\gamma}} = (1-\xi^2)^{\frac{1}{\gamma}} < 1,$$

which contradicts (31).

It follows that c_t converges to a positive number, and so does $u'(c_t)$. Since k_t is bounded, (24) clearly holds. □

Now let us consider different cases that may arise. If $\delta f'(k_{t+1}) \leq 1$ for all sufficiently large t, then the transversality condition (24) holds by Lemma 3.

Suppose there exists Θ such that $\delta f'(k_{\Theta+1}) > 1$. Then there are only two possibilities. Either there exists $\Theta_1 > \Theta$ such that $\delta f'(k_{\Theta_1+1}) \leq 1$ or $\delta f'(k_{t+1}) > 1$ for all $t > \Theta$. In the former case, either $\delta f'(k_{t+1}) \leq 1$ for all $t > \Theta_1$ so that Lemma 3 holds or there exists $\Theta_2 > \Theta_1$ such that $\delta f'(k_{\Theta_2+1}) > 1$ in which case we are in the conditions of Lemma 2. In the latter case, for all $t > \Theta$, $u'(c_t) < u'(c_\Theta)$ and $k_{t+1} \leq (f')^{-1}(1/\delta)$. Therefore, $\delta^t u'(c_t)k_{t+1} < \delta^t u'(c_\Theta)(f')^{-1}(1/\delta)$, which implies (24).

B.2.2 Case 2 (linear production function)

The first-order conditions in this case state that $A^t \delta^t c_t^{-\rho_i} = c_0^{-\rho_i}$, $t = 1, 2, \ldots$.

Assume that the transversality condition (24) fails. Then there exist Θ and $N > 0$ such that $\delta^t c_t^{-\rho_i} k_{t+1} \geq N$, for all $t > \Theta$, and

$$\frac{k_{t+1}}{A^t} \geq \frac{N}{c_0^{-\rho_i}}, \quad t > \Theta.$$

Hence for all $t > \Theta$,

$$e_{t+1} = \frac{c_{t+1}}{Ak_{t+1}} \leq \frac{c_0^{-\rho_i}}{N}\frac{c_{t+1}}{A^{t+1}}.$$

Therefore, by (30),

$$\sum_{t=\Theta}^{\infty} \frac{c_{t+1}}{A^{t+1}} = +\infty.$$

However, iterating the equation $c_t + k_{t+1} = Ak_t$, we easily get

$$c_0 + \frac{c_1}{A} + \frac{c_2}{A^2} + \ldots \leq Ak_0,$$

a contradiction.

C Proof of Propositions 2 and 3

Consider agent i with the discount factor δ_i and the felicity function $u_i(c)$. Note that for all τ (except $\tau = T$ in the finite horizon case) and any non-degenerate expectations $E_{\tau+1,T}$, her preferred time τ consumption rate is a unique solution to the following equation:

$$\frac{\partial V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})}{\partial e_\tau} = 0, \quad 0 \le \tau < T.$$

Indeed, the above equation can be rewritten as

$$\Phi_\tau^i(k_\tau, e_\tau) = \Psi_\tau^i(k_\tau, e_\tau, E_{\tau+1,T}). \tag{32}$$

where $\Phi_\tau^i(k_\tau, e_\tau)$ is defined by (20), and $\Psi_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})$ is defined by (21). It follows from Remark 1 that there exists a unique solution e_τ^* to equation (32), and $0 < e_\tau^* < 1$. Thus there is a unique time τ preferred consumption rate for agent i.

Using (19) and Remark 1, we infer that $\frac{\partial V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})}{\partial e_\tau}$ is strictly decreasing in e_τ, and thus the preferences of agent i in voting over the time τ consumption rate are strictly concave. Thus the preferences of each agent in voting over every consumption rate are single-peaked, which proves Proposition 2.

Now suppose that all agents have the same felicity function $u(c)$ and the same non-degenerate expectations $E_{\tau+1,T}$. Then for all τ (except $\tau = T$ in the finite horizon case) higher values of the discount factor δ_i correspond to lower values of the preferred time τ consumption rate e_τ^{i*}. Indeed, $\Phi_\tau^i(k_\tau, e_\tau)$ is independent of δ_i and strictly decreasing in e_τ, while $\Psi_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})$ is strictly increasing both in δ_i and e_τ. Therefore, the solution to equation (32), e_τ^*, is strictly decreasing in δ_i.

It follows from the median voter theorem that the time τ voting equilibrium is the preferred consumption rate for the agent with the median discount factor, which proves Proposition 3.

D Proof of Theorem 2

Consider the constant sequence of consumption rates $E^* = \{e^*, e^*, \ldots\}$, where e^* is given by (11). Consider a fictitious agent with the discount factor δ_{med} and felicity function $u_i(c)$, i.e., the agent with the median discount factor and the felicity function of agent i. It is clear that e^* is the preferred time τ consumption rate for this agent, and E^* is her optimum in terms of consumption rates.

Let τ be an arbitrary point in time. Suppose that expectations of agents are constant and equal to $E_{\tau+1,\infty} = \{e^*, e^*, \ldots\}$. It follows from Proposition 2 that the preferences of each agent in voting over the time τ consumption rate are strictly concave. Consider agent i with the discount factor δ_i and the felicity function $u_i(c)$. She has the unique preferred time τ consumption rate e_τ^{i*}. It follows from the results of Appendix C that if $\delta_i > \delta_{med}$ ($\delta_i < \delta_{med}$) then $e_\tau^{i*} < e^*$ ($e_\tau^{i*} > e^*$).

Hence the winner in voting over the time τ consumption rate under constant expectations $E_{\tau+1,\infty} = \{e^*, e^*, \ldots\}$ is precisely e^* given by (11). Since the point in time τ is chosen arbitrarily, e^* is the winner in voting over each consumption rate under the expectations $\{e^*, e^*, \ldots\}$. Thus the sequence $\{e^*, e^*, \ldots\}$ is an intertemporal voting equilibrium, and hence e^* is a stationary voting equilibrium.

Let us now prove that e^* is the unique stationary voting equilibrium. Suppose that there is another stationary voting equilibrium $\tilde{e}$. Consider an arbitrary point in time τ. Suppose that the expectations of agents are constant and equal to $E_{\tau+1,\infty} = \tilde{E} = \{\tilde{e}, \tilde{e}, \ldots\}$. It follows from Proposition 2 that the preferences of each agent in voting over the time τ consumption rate are strictly concave. By the median voter theorem, $\tilde{e}$ is the most preferred consumption rate for some "median" voter. Clearly, $\tilde{e}$ is the preferred consumption rate for this same agent for all $\tau = 0, 1, \ldots$. Therefore, the sequence $\{\tilde{e}, \tilde{e}, \ldots\}$ is a step-by-step intertemporal optimum for this agent.

Denote the discount factor of this agent by $\tilde{\delta}$. Consider the corresponding $\tilde{k}$, which is the unique positive solution to the equation $k = (1 - \tilde{e})f(k)$. Clearly, $\tilde{k}$ is determined by the "modified golden rule" for this agent: $\tilde{\delta} f'(\tilde{k}) = 1$. It follows that $\tilde{e}$ depends only on $\tilde{\delta}$, and is independent of the felicity function of this agent. In other words, $\tilde{e}$ is the preferred time τ consumption rate for the fictitious agent with the discount factor $\tilde{\delta}$ and any felicity function, in voting over e_τ given $k_\tau = \tilde{k}$ and the expectations $\tilde{E}$.

Now suppose that $\tilde{e} > e^*$ ($\tilde{e} < e^*$), and thus $\tilde{\delta} < \delta_{med}$ ($\tilde{\delta} > \delta_{med}$). Consider agent i with the discount factor $\delta_i \ge \delta_{med}$ ($\delta_i \le \delta_{med}$) and the felicity function $u_i(c)$. Since $\delta_i > \tilde{\delta}$ ($\delta_i < \tilde{\delta}$), it follows from the results of Appendix C that $e_\tau^{i*} < \tilde{e}$ ($e_\tau^{i*} > \tilde{e}$). Hence for at least $\frac{N+1}{2}$ agents their preferred time τ consumption rates are are lower (resp. greater) than $\tilde{e}$. It follows that $\tilde{e}$ is not a Condorcet winner in voting over the time τ consumption rate under expectations $\tilde{E}$, and cannot be a stationary voting equilibrium. Thus e^* is the unique stationary voting equilibrium.

E Proof of Theorem 3

Let τ be an arbitrary point in time. Consider how agents vote over the time τ consumption rate under constant non-degenerate expectations $E_{\tau+1,\infty} = \{e, e, \ldots\}$.

The objective function of agent i in the time τ voting problem under these expectations is given by:

$$V_\tau^i(k_\tau, e_\tau, E_{\tau+1,\infty}) = u_i(e_\tau A k_\tau) + \delta_i u_i\left(e(1-e_\tau)A^2 k_\tau\right) + \delta_i^2 u_i\left(e(1-e)(1-e_\tau)A^3 k_\tau\right) + \ldots.$$

It follows from Proposition 2 that preferences of agent i are concave, and her preferred time τ consumption rate, e_τ^{i*}, is the unique solution to the following equation:

$$\frac{\partial V_\tau^i(k_\tau, e_\tau, E_{\tau+1,\infty})}{\partial e_\tau} = 0,$$

which can be rewritten as

$$Ak_\tau (e_\tau A k_\tau)^{-\rho_i} = \delta_i e A^2 k_\tau \left(e(1-e_\tau)A^2 k_\tau\right)^{-\rho_i} + \delta_i^2 e(1-e)A^3 k_\tau \left(e(1-e)(1-e_\tau)A^3 k_\tau\right)^{-\rho_i} + \ldots.$$

Dividing both parts of the above equation by $(Ak_\tau)^{1-\rho_i}$, we get

$$(e_\tau)^{-\rho_i} = (1-e_\tau)^{-\rho_i}\left(A\delta_i e(Ae)^{-\rho_i} + A^2\delta_i^2 e(1-e)(A^2 e(1-e))^{-\rho_i} + \ldots\right),$$

and hence, taking into account that expectations $\{e, e, \ldots\}$ are non-degenerate, we obtain

$$\left(\frac{1-e_\tau}{e_\tau}\right)^{\rho_i} = A\delta_i\left(e(Ae)^{-\rho_i} + A\delta_i e(1-e)\left(A^2 e(1-e)\right)^{-\rho_i} + \ldots\right) = \frac{\delta_i (Ae)^{1-\rho_i}}{1-\delta_i (A(1-e))^{1-\rho_i}}.$$

Using (12) and the above equation, we conclude that the preferred time τ consumption rate for agent i is the solution to the following equation in e_τ:

$$\left(\frac{1-e_\tau}{e_\tau}\right)^{\rho_i} = \frac{\left(\frac{1+\gamma_i}{A}\right)^{\rho_i} e^{1-\rho_i}}{1-\left(\frac{1+\gamma_i}{A}\right)^{\rho_i}(1-e)^{1-\rho_i}}. \tag{33}$$

Note that the preferred time τ consumption rate for each agent is independent of the current capital stock k_τ, and depends only on constant expectations e.

Denote the solution to equation (33) depending on e by $e_\tau^{i*}(e)$.

Lemma 4. *If* $1+\gamma_i \gtreqqless A(1-e)$*, then* $e_\tau^{i*}(e) \lesseqqgtr e$.

Proof. Suppose that $1+\gamma_i \gtreqqless A(1-e)$. Since $e_\tau^{i*}(e)$ is the solution to equation (33), we have

$$\left(\frac{1-e_\tau}{e_\tau}\right)^{\rho_i} \gtreqqless \frac{(1-e)^{\rho_i} e^{1-\rho_i}}{1-(1-e)^{\rho_i}(1-e)^{1-\rho_i}} = \frac{(1-e)^{\rho_i} e^{1-\rho_i}}{e} = \left(\frac{1-e}{e}\right)^{\rho_i}.$$

The expression $\frac{1-e}{e}$ is decreasing in e for $0 < e < 1$. Therefore, $e_\tau^{i*} \lesseqqgtr e$. □

Let e^* be given by (13). Then

$$1+\gamma_{med} = A(1-e^*).$$

If the expectations are $\{e^*, e^*, \ldots\}$, then, by Lemma 4,

$$\gamma_i \gtreqqless \gamma_{med} \Rightarrow e_\tau^{i*} \lesseqqgtr e^*.$$

Therefore, for any time τ, e^* is the Condorcet winner in voting over the time τ consumption rate under the expectations $\{e^*, e^*, \ldots\}$. It follows that e^* is a balanced-growth voting equilibrium.

Moreover, any other consumption rate $\tilde{e}$ cannot be a balanced-growth voting equilibrium. It follows from Lemma 4 that if $\tilde{e} > e^*$ ($\tilde{e} < e^*$), then $A(1-\tilde{e}) < 1+\gamma_{med}$ ($A(1-\tilde{e}) > 1+\gamma_{med}$), and for at least $\frac{N+1}{2}$ agents their preferred time τ consumption rates in voting over the time τ consumption rate under expectations $\{\tilde{e}, \tilde{e}, \ldots\}$ are lower (resp. greater) than $\tilde{e}$. Hence $\tilde{e}$ is not a Condorcet winner, and is not a balanced-growth voting equilibrium.

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